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Down–Up Algebras and Ambiskew Polynomial Rings

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We show that the down–up algebras of G. Benkart (1998, *in* "Recent Progress in Algebra," Contemporary Mathematics Vol. 224, Am. Math. Soc., Providence) and G. Benkart and T. Roby (1998, *J. Algebra* **209**, 305–344) lie in a certain class of iterated skew polynomial rings, called ambiskew polynomial rings, in two indeterminates x and y over a commutative ring B. In such rings, commutation of the indeterminates with elements of B involve the same endomorphism σ of B, but from different sides, that is, $yb = \sigma(b)y$ and $bx = x\sigma(b)$, and, for some scalar p, $yx - pxy \in B$. In previous studies of ambiskew polynomial rings, σ was required to be an automorphism but, in order to cover all down–up algebras, this requirement must be dropped. The Noetherian down–up algebras are those where σ is an automorphism and, in this case, we apply existing results on ambiskew polynomial rings to determine the finite-dimensional simple modules and the prime ideals. We adapt the methods underlying these results so as to apply to the non-Noetherian down–up algebras for which they reveal a surprisingly rich structure.

Key Words: skew polynomial ring; down–up algebra.

1. INTRODUCTION

The down–up algebras $A(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$, were introduced by Benkart and Roby [7, 8] as generalizations of algebras generated by a pair of operators, the "down" and "up" operators, acting on the vector space $\mathbb{C}P$ for certain partially ordered sets P. Examples include the enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and deformations of it due to Woronowicz [35] and Witten [34]. Kirkman et al. [25] have shown that, for a down–up algebra $A(\alpha, \beta, \gamma)$ to be Noetherian, it is necessary and sufficient for β to be non-zero. They offered two proofs for sufficiency, one presenting $A(\alpha, \beta, \gamma)$ as a generalized Weyl algebra in the sense of Bavula [1–4] and

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one showing that $A(\alpha, \beta, \gamma)$ has a filtration for which the associated graded ring is an iterated skew polynomial ring over $\mathbb{C}$.

The examples mentioned above are basic examples in a certain class of iterated skew polynomial rings $R(B, \sigma, c, p)$, for which we now propose the name *ambiskew polynomial ring*, studied, at various levels of generality, in a sequence of papers [15–17, 19–21]. Of these, the paper whose level of generality is closest to being appropriate for this paper is [21], where K is a field, σ is a K-automorphism of a commutative K-algebra B, $c \in B$, and $p \in K \setminus \{0\}$. However, here we shall not always require σ to be bijective and p to be non-zero. Thus σ will be an arbitrary K-endomorphism of B and $p \in K$. To form $R(B, \sigma, c, p)$, two indeterminates x and y are adjoined to B with $yb = \sigma(b)y$ and $bx = x\sigma(b)$, for all $b \in B$, and with $yx - pxy = c$. Ambiskew polynomial rings are closely related to generalized Weyl algebras in that every generalized Weyl algebra is isomorphic to a factor of an ambiskew polynomial ring [16] and every ambiskew polynomial ring $R(B, \sigma, c, p)$ with σ bijective and $p \neq 0$ can be presented as a generalized Weyl algebra over the polynomial ring $B[w]$ [4, 21].

A substantial part of [8] is devoted to the development of a theory of Verma modules generalizing that for $U(\mathfrak{sl}_2(\mathbb{C}))$. Such a theory is also a vital aspect in the study of ambiskew polynomial rings. The basic examples, the Verma module theory, and the connection with generalized Weyl algebras suggest a strong connection between down–up algebras and ambiskew polynomial rings. We shall see, in Section 7, that the down–up algebras are precisely the ambiskew polynomial rings over $\mathbb{C}[t]$ in which $\deg c = 1$ and $\deg \sigma(t) \leq 1$. In Sections 2 to 6, we shall work only with ambiskew polynomial rings in which σ is bijective and $p \neq 0$. In Section 3, we shall see that the Noetherian down–up algebras are precisely the ambiskew polynomial rings over $\mathbb{C}[t]$ in which $\deg c = 1$, $p \neq 0$, and σ is bijective.

The presentations of Noetherian down–up algebras as generalized Weyl algebras arising from our presentations of them as ambiskew polynomial rings are different to those in [25] and involve triangular automorphisms in the sense of [28]. In [18] we gave a primitivity criterion for generalized Weyl algebras and applied it to generalized Weyl algebras over $\mathbb{C}[t, w]$, where the resulting primitivity criterion is in terms of triangular automorphisms. Consequently we obtain, in Section 4, a primitivity criterion for Noetherian down–up algebras.

When $\beta \neq 0$, the results and methods of [16, 19, 21] can be used to classify the finite-dimensional simple representations of down–up algebras and to determine when all finite-dimensional representations are semisimple. This is done in Section 5. In some cases the only obstruction to all finite-dimensional simple modules being semisimple is the existence of a non-Artinian commutative factor and this obstruction can be removed by localization at the powers of a certain normal element.

The results of [17] determine the height one prime ideals when the defining automorphism σ of $\mathbb{C}[t]$ has infinite order and, as the factors by these prime ideals are familiar algebras, this leads to a complete description of the prime spectrum, given in Section 6.

Sections 7, 8, and 9 are concerned with the non-Noetherian case, $\beta = 0$, and with the extension of the definition of ambiskew polynomial ring needed to accommodate it. Although some of the general techniques, in particular localization at powers of the indeterminates, which have been useful when σ is bijective and $p \neq 0$, are no longer helpful, others can be adapted to derive information on the non-Noetherian down–up algebras. In particular, in Section 8, we classify the finite-dimensional simple modules and, in Section 9, we determine the prime spectrum when either α is non-zero and not a root of unity, or $\alpha = 1$ and $\gamma \neq 0$. In the latter case there is a unique d-dimensional simple module for each positive integer d and the prime spectrum turns out to have some properties analogous to those of the prime spectrum of $U(\mathfrak{sl}_2(\mathbb{C}))$, though the analogy is not exact. In particular, there is a family of height one prime ideals P_η, where $\eta \in \mathbb{C}\backslash C$ for a countable set C, which are annihilators of Verma modules and are such that $A(\alpha, 0, \gamma)/P_\eta$ has a unique minimal non-zero ideal Q/P_η. This ideal is prime and the height two prime ideal Q of $A(\alpha, 0, \gamma)$ is independent of η. The factor $A(\alpha, 0, \gamma)/Q$ is isomorphic to the Weyl algebra A_1.

Our methods are based on those in [16, 17, 19, 21] but there are inevitable aspects in common with ideas and methods used in existing papers on down–up algebras, in particular [7–9, 25, 26]. Also some of our results, in particular those on representations, may be approached through those of [1–4] on generalized Weyl algebras, though we have not checked the details.

Down–up algebras were originally defined over $\mathbb{C}$ but can be defined over an arbitrary field. Some papers take the latter approach, as did earlier versions of this paper. For reasons of economy we shall consider only down–up algebras over $\mathbb{C}$. However, much of what we say is valid more generally, though care needs to be taken with characteristic, algebraic closure and, occasionally, countability.

2. PRELIMINARIES

2.1. *Down–Up Algebras*

For arbitrary $\alpha, \beta, \gamma \in \mathbb{C}$, the *down–up algebra* $A(\alpha, \beta, \gamma)$ [7, 8] is the $\mathbb{C}$-algebra generated by d and u subject to the relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma d, \tag{1a}$$

$$du^2 = \alpha udu + \beta u^2d + \gamma u. \tag{1b}$$

The motivation for studying such algebras and the reason for their name come from combinatorics and are discussed in [8, Sect. 1]. We give some brief details here. Let P be a partially ordered set and let $\mathbb{C}P$ be the complex vector space with basis P. If, for each $p \in P$, the set $\{x \in P : x \succ p\}$ of successors of p and the set $\{x \in P : x \prec p\}$ are finite then we can define the "down" operator d and the "up" operator u in $\mathrm{End}_{\mathbb{C}}\, \mathbb{C}P$ as follows: $u(p) = \sum_{x \succ p} x$ and $d(p) = \sum_{x \prec p} x$. For partially ordered sets in general one needs to complete $\mathbb{C}P$ in order to define d and u. The relations satisfied by d and u may have significant structural implications for P. A partially ordered set P is called (q, r)-*differential* if there exist $q, r \in \mathbb{C}$ such that the down and up operators for P satisfy (1a) and (1b) when $\alpha = q(q+1)$, $\beta = -q^3$ and $\gamma = r$. Four significant examples of such partially ordered sets, in which q is a prime power, are discussed in [8, Sect. 1; 32]. Here, for $q, r \in \mathbb{C}$, we shall say that the down–up algebra $A(\alpha, \beta, \gamma)$ is (q, r)-*differential* if $\alpha = q(q+1)$, $\beta = -q^3$, and $\gamma = r$ and we shall regularly illustrate results by referring to this case.

It is observed in [8] that, for $0 \neq \lambda \in \mathbb{C}$, $A(\alpha, \beta, \gamma) \simeq A(\alpha, \beta, \lambda\gamma)$. Consequently, when considering cases where $\gamma \neq 0$, there will be no loss of generality in assuming that $\gamma = 1$.

2.2. *Ambiskew Polynomial Rings*

Here we present the details of ambiskew polynomial rings in the generality appropriate to down–up algebras with $\beta \neq 0$, namely in the generality of [21]. Let K be a field and let K^* denote the multiplicative group $K\backslash\{0\}$. Let B be a commutative K-algebra, let σ be a K-automorphism of B, let $c \in B$, and let $p \in K^*$. Let S be the skew polynomial ring $B[x; \sigma^{-1}]$ and extend σ to S by setting $\sigma(x) = px$. By [10, 0.8, p. 41] or [13, Exercise 1F], there is a σ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$. The *ambiskew polynomial ring* $R = R(B, \sigma, c, p)$ is the skew polynomial ring $S[y; \sigma, \delta]$. Thus

$$yx - pxy = c \tag{2}$$

and, for all $b \in B$,

$$xb = \sigma^{-1}(b)x \qquad \text{and} \qquad yb = \sigma(b)y.$$

Alternatively, $R = B[y; \sigma][x; \sigma^{-1}, \delta']$, where $\sigma(y) = p^{-1}y$, $\delta'(B) = 0$, and $\delta'(y) = -p^{-1}c$ so that $xy - p^{-1}yx = -p^{-1}c$, which is equivalent to (2). The latter presentation is closer to that in [17, 19, 21] where $p^{-1} = \rho$ and $-p^{-1}c = v$. Here we favour the former presentation as later we will relax the condition that $p \neq 0$.

Rewriting the relation $xb = \sigma^{-1}(b)x$ as $bx = x\sigma(b)$, we see that the construction involves twists from both sides using σ. This is both the reason for the name and the key to generalizing the construction to the case where σ is not bijective.

Ambiskew polynomial rings are closely related to the generalized Weyl algebras studied in [1–4]. Given an automorphism σ and a central element a of a ring B, the generalized Weyl algebra $B(\sigma, a)$ is the ring extension of B generated by X^- and X^+ subject to the relations

$$X^-X^+ = a, \qquad X^+X^- = \sigma(a) \tag{3a}$$

and, for all $b \in B$,

$$X^+b = \sigma(b)X^+, \qquad X^-\sigma(b) = bX^-. \tag{3b}$$

PROPOSITION 2.1. *The ambiskew polynomial ring $R(B, \sigma, c, p)$ is isomorphic to the generalized Weyl algebra $B[w](\sigma, w)$, where σ is extended to $B[w]$ by setting $\sigma(w) = pw + \sigma(c)$. In the conformal case, $R(B, \sigma, c, p) \simeq B[z](\sigma, z + \sigma(a))$, where $\sigma(z) = pz$.*

Proof. See [21, 2.6 Corollary] or [4, Lemma 1.2]. ∎

2.3. *Casimir Elements*

If $c = \sigma(a) - pa$ for some $a \in B$ then, as in [21], we shall say that the four-tuple (B, σ, c, p) is *conformal*. Suppose that this is the case and let $z = yx - \sigma(a) = p(xy - a)$. Then,

$$yz = py(xy - a) = p(yxy - \sigma(a)y) = pzy, \tag{4a}$$

$$zx = p(xy - a)x = p(xyx - x\sigma(a)) = pxz, \tag{4b}$$

and, for all $b \in B$,

$$zb = p(x\sigma(b)y - ab) = bz. \tag{4c}$$

Thus z is a normal element of R, which we call the *Casimir element* of R, and induces a K-automorphism ζ of R such that $\zeta(b) = b$ for all $b \in B$, $\zeta(y) = p^{-1}y$, and $\zeta(x) = px$. If $p = 1$ then z is central.

In the general case, we set $w = yx = pxy + c$, noting that $bw = wb$ for all $b \in B$ and that, in the conformal case, $w = z + \sigma(a)$ and $B[z] = B[w]$.

We extend σ to a K-automorphism, also denoted σ, of $B[w]$ by setting $\sigma(w) = pw + \sigma(c)$ and $\sigma^{-1}(w) = p^{-1}(w - c))$. Then $yw = \sigma(w)y$ and $xw = \sigma^{-1}(w)x$. In the conformal case, $\sigma(z) = pz$.

2.4. *Skew Commutator Formulae*

Let $R(B, \sigma, c, p)$ be an ambiskew polynomial ring. For $d \geq 1$, let

$$c_d = \sum_{j=0}^{d-1} p^{d-j-1}\sigma^j(c). \tag{5}$$

In the conformal case, where $c = \sigma(a) - pa$, $c_d = \sigma^d(a) - p^d a$. The following "skew commutator formulae" from [21, 2.3], which hold for $d \geq 1$ and are routinely checked by induction, are very effective in the study of ambiskew polynomial rings:

$$yx^d - p^d x^d y = x^{d-1}c_d; \tag{6a}$$

$$y^d x - p^d xy^d = c_d y^{d-1}. \tag{6b}$$

From (6a), it follows that, for $n \geq 1$,

$$y^{n+1}x^{n+1} = y^n(p^{n+1}x^{n+1}y + x^n c_{n+1}) = y^n x^n(p^{n+1}xy + c_{n+1}),$$

from which it follows, inductively, that

$$y^d x^d = \prod_{i=1}^{d}(p^i xy + c_i). \tag{7}$$

In the conformal case, this becomes

$$y^d x^d = \prod_{i=1}^{d}(p^{i-1}z + \sigma^i(a)). \tag{8}$$

Suppose that (B, σ, c, p) is conformal. The factor ring R/zR is the ring extension of B generated by X^- and X^+, the images of x and y, respectively, subject to the relations

$$X^-X^+ = a, \qquad X^+X^- = \sigma(a)$$

and, for all $b \in B$,

$$X^+b = \sigma(b)X^+, \qquad X^-\sigma(b) = bX^-.$$

Thus R/zR is the generalized Weyl algebra $B(\sigma, a)$.

2.5. *Notation*

Let $q \in K$ and let $n \geq k$ be non-negative integers. Then we define $[n]_q := \sum_{j=0}^{n-1} q^j$. Thus $[n]_q = (q^n - 1)/(q-1)$ if $q \neq 1$. The *q-binomial coefficient* $\left[{n \atop k}\right]_q$ is defined to be

$$\frac{[n]!_q}{[k]!_q[n-k]!_q},$$

where $[n]!_q = [n]_q[n-1]_q \cdots [1]_q$. The following is a consequence of [22, Proposition IV.2.7].

PROPOSITION 2.2. *For τ, $\psi \in K$ and each positive integer n,*

$$(\tau + \psi)(\tau + q\psi)\cdots(\tau + q^{n-1}\psi) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \tau^{n-k} \psi^k. \tag{9}$$

2.6. *Quantized Weyl Algebras and Quantum Coordinate Rings*

Prime factors of $A(\alpha, \beta, \gamma)$ can include quantized Weyl algebras and coordinate rings of quantum planes. Our notation for these will be as follows. For $q \in \mathbb{C}^*$, the *quantized Weyl algebra* A_1^q is the $\mathbb{C}$-algebra generated by x and y subject to the relation $xy - qyx = 1$ and the *coordinate ring $C(q)$ of the quantum plane* is the $\mathbb{C}$-algebra generated by x and y subject to the relation $xy = qyx$.

When $q \neq 1$, A_1^q has a normal element $z = xy - yx$, with $A_1^q/zA_1^q \simeq \mathbb{C}[y^{\pm 1}]$, and when q is not a root of unity, the localization of A_1^q at $\{z^i\}_{\{i \geq 1\}}$ is simple, whence zA_1^q is the unique height one prime of A_1^q and the other non-zero prime ideals have the form $zA_1^q + (y - \tau)A_1^q$, $\tau \in \mathbb{C}^*$. For example, see [12, 8.4].

In $C(q)$, x and y are normal, $C(q)/xC(q) \simeq \mathbb{C}[y]$, and $C(q)/yC(q) \simeq \mathbb{C}[x]$. When q is not a root of unity, by [29, 1.8.7(ii)], the localization of $C(q)$ at $\{x^i y^j\}_{\{i,j \geq 1\}}$ is simple and $xC(q)$ and $yC(q)$ are the only height one primes of $C(q)$.

To classify the finite-dimensional simple modules over a down–up algebra, one needs to know the classification of such modules over A_1^q and $C(q)$. This is summarised below.

Let $T = A_1^q$. If $q = 1$ then, as is well known, there are no non-zero finite-dimensional T-modules. If q is not a root of unity then, by [19, Example 4.1], the finite-dimensional simple T-modules are one-dimensional of the form $T/((y - \xi)T + (x - \frac{1}{\xi(1-q)})T)$, where $\xi \in \mathbb{C}^*$. If q is a primitive nth root of unity for some $n > 1$ then, by [19, Example 4.1], the finite-dimensional simple T-modules are one-dimensional of the form $T/((y - \xi)T + (x - \frac{1}{\xi(1-q)})T)$ or n-dimensional and either of the form $T/(xT + y^n T)$

or of one of overlapping forms $T/((xy-\eta)T+(y^n-\xi)T)$ and $T/(yx-\eta)T+(x^n-\xi)T)$, where $\eta,\xi\in\mathbb{C}$ with $\xi\neq 0$.

Now let $T=C(q)$. If $q=1$ then $C(q)=\mathbb{C}[x,y]$ is commutative and all simple finite-dimensional modules are one-dimensional. If q is not a root of unity then every simple finite-dimensional T-module is one-dimensional of the form $T/((y-\xi)T+(x-\psi)T)$ for some $\xi,\psi\in\mathbb{C}$ with $\xi\psi=0$. This is not difficult to see but can be deduced from [19, 3.7].

If q is a primitive nth root of unity, with $n>1$, then it can be deduced from [19, 3.7] that the finite-dimensional simple T-modules are one-dimensional of the form $T/((y-\xi)T+(x-\psi)T)$ or n-dimensional of the form $T/((xy-\eta)T+(y^n-\psi)T)$, where $\eta,\psi\in\mathbb{C}^*$.

3. $A(\alpha,\beta,\gamma)$ AS AN AMBISKEW POLYNOMIAL RING

Let $\alpha,\beta,\gamma\in\mathbb{C}$. Throughout this section, we assume that $\beta\neq 0$. Let μ_1 and μ_2 be the roots, necessarily non-zero, of the equation

$$\beta X^2+\alpha X-1=0 \tag{10}$$

and let H be the subgroup $\langle\mu_1,\mu_2\rangle$ of $\mathbb{C}^*$. We shall present $A(\alpha,\beta,\gamma)$ as an ambiskew polynomial ring over the polynomial ring $\mathbb{C}[t]$ with either $\sigma(t)=\lambda t$ for some $\lambda\in\mathbb{C}^*$, or $\sigma(t)=t+\tau$ for some $\tau\in\mathbb{C}$, and use Proposition 2.1 to present $A(\alpha,\beta,\gamma)$ as a generalized Weyl algebra over $\mathbb{C}[t,z]$. The defining automorphism σ of $\mathbb{C}[t,z]$ will always be of one of the following three types

$$z\mapsto\lambda z,\qquad t\mapsto\mu t\qquad(\lambda,\mu\in\mathbb{C}^*); \tag{11a}$$

$$z\mapsto\lambda z,\qquad t\mapsto t+\nu\qquad(\lambda,\nu\in\mathbb{C}^*); \tag{11b}$$

$$z\mapsto\lambda z+\sum_{\lambda=\mu^i}\eta_i t^i,\qquad t\mapsto\mu t\qquad(\lambda,\mu\in\mathbb{C}^*,\ \eta_i\in\mathbb{C}). \tag{11c}$$

These are *triangular* automorphisms [28] and, by [30, Proposition 1], every triangular automorphism is conjugate to an automorphism of one of the three listed types.

There will be four cases of down–up algebras to consider, depending on the nature of μ_1 and μ_2. The same four cases have been identified in [8].

3.1. *Case* A

Here we assume that $\mu_1\neq\mu_2$ and that $\mu_i\neq 1$ for $i=1,2$. Equivalently, $\alpha^2+4\beta\neq 0$ and $\alpha+\beta\neq 1$.

Let $B = \mathbb{C}[t]$, let σ be the $\mathbb{C}$-automorphism of $\mathbb{C}[t]$ such that $\sigma(t) = \mu_2^{-1}t$, let $p = \mu_1^{-1}$, and let $c = -\mu_1^{-1}(t + (\mu_1\mu_2\gamma/(1-\mu_2)))$. By [11, Proposition 1], $R(B, \sigma, c, p)$ is generated by x, y and t subject to three relations

$$\mu_2 yt = ty, \tag{12a}$$

$$xt = \mu_2 tx, \tag{12b}$$

$$xy - \mu_1 yx = t + \frac{\mu_1\mu_2\gamma}{1-\mu_2}. \tag{12c}$$

By (12c), the generator t is redundant. It is routine to use (12c) to substitute for t in (12a) and (12b) and deduce that x and y satisfy the down–up relations (1a) and (1b), with $y = d$ and $x = u$, and, conversely, to check that if two generators x, y satisfy (1a) and (1b), and (12c) is used to define t, then (12a) and (12b) hold. Therefore $A(\alpha, \beta, \gamma) = R(\mathbb{C}[t], \sigma, c, p)$.

Here $(\mathbb{C}[t], \sigma, c, p)$ is conformal with $a = (\mu_2/(\mu_2 - \mu_1))t + (\gamma\mu_1\mu_2/(1-\mu_1)(1-\mu_2))$. Thus there is a Casimir element

$$z = yx - \frac{1}{\mu_2 - \mu_1}t - \frac{\gamma\mu_1\mu_2}{(1-\mu_1)(1-\mu_2)}$$

and, by Proposition 2.1,

$$A(\alpha, \beta, \gamma) \simeq k[t, z](\sigma, z + \sigma(a)), \qquad \sigma(t) = \mu_2^{-1}t,\ \sigma(z) = \mu_1^{-1}z. \tag{13}$$

By symmetry, $A(\alpha, \beta, \gamma)$ has a second presentation as an ambiskew polynomial ring, namely $R(\mathbb{C}[z], \sigma, c, \mu_2)$, where $\sigma(z) = \mu_1^{-1}z$ and $c = z + (\mu_1\mu_2\gamma/(1-\mu_1))$. The eigenvector for σ in one case becomes the Casimir element in the other. One application of this simple observation is in showing that certain deformations of $U(\mathfrak{sl}_2(\mathbb{C}))$ given by Woronowicz [35] and Witten [34] are isomorphic. Details of this will appear in [5].

3.2. *Case* B

Here we assume that $\mu_1 = 1 \neq \frac{-1}{\beta} = \mu_2$ or, equivalently, $\alpha + \beta = 1$ and $\alpha \neq 2$.

Let σ be the $\mathbb{C}$-automorphism of $\mathbb{C}[t]$ such that $\sigma(t) = -\beta t$, let $p = 1$, and let $c = -t + \frac{\gamma}{\beta+1}$. Then $R(B, \sigma, c, p)$ is generated by x, y and t subject to the relations

$$yt = -\beta ty, \tag{14a}$$

$$-\beta xt = tx, \tag{14b}$$

$$xy - yx = t - \frac{\gamma}{\beta+1}. \tag{14c}$$

These are the same as in Case A, with $\mu_1 = 1$ and $\mu_2 = -1/\beta$, so again $A(\alpha, \beta, \gamma) = R(\mathbb{C}[t], \sigma, c, p)$. Whether $(\mathbb{C}[t], \sigma, c, p)$ is conformal here depends on whether $\gamma = 0$. First suppose that $\gamma = 0$. Then $(\mathbb{C}[t], \sigma, c, p)$ is conformal with $a = \frac{1}{1+\beta}t$, the Casimir element is $z = yx + \frac{\beta}{1+\beta}t = xy - \frac{1}{1+\beta}t$, and $yx + \beta xy = (1+\beta)z$. By Proposition 2.1,

$$A(\alpha, \beta, \gamma) \simeq \mathbb{C}[t, z](\sigma, z + \sigma(a)), \qquad \sigma(t) = -\beta t, \sigma(z) = z. \tag{15}$$

Again σ is triangular of type (a) and there is an alternative presentation of $A(\alpha, \beta, \gamma)$ as an ambiskew polynomial ring, namely $R(\mathbb{C}[z], \sigma, c, p)$, where

$$\sigma(z) = z, p = -\beta \qquad \text{and} \qquad c = (1+\beta)z. \tag{16}$$

Now suppose that $\gamma \neq 0$. Without loss of generality, $\gamma = 1$. Note that, for $a \in \mathbb{C}[t]$, $\sigma(a) - a$ has zero constant term so $(\mathbb{C}[t], \sigma, c, p)$ is not conformal. Let $f = (\beta + 1)w + \beta t - 1$. Then $\mathbb{C}[t, w] = \mathbb{C}[t, f]$ and $\sigma(f) = f + 1$. By Proposition 2.1,

$$A(\alpha, \beta, \gamma) \simeq \mathbb{C}[t, f](\sigma, w), \quad \sigma(t) = -\beta t, \sigma(f) = f + 1. \tag{17}$$

Here σ is triangular of type (b). Although there is no symmetry and no Casimir element, we again get an alternative presentation of $A(\alpha, \beta, \gamma)$ as an ambiskew polynomial ring. Let

$$h = \beta^{-1}(f + (\beta+1)^{-1}) = \beta^{-1}((\beta+1)yx + \beta t - \beta/(1+\beta)).$$

Using (14c) to substitute for t, we see that $h = xy + \beta^{-1}yx$. Also $\sigma(h) = h + \beta^{-1}$ so h, x, and y satisfy the relations

$$hy - yh = -\beta^{-1}y, \tag{18a}$$

$$hx - xh = \beta^{-1}x, \tag{18b}$$

$$yx + \beta xy = \beta h \tag{18c}$$

and

$$A(\alpha, \beta, \gamma) = R(\mathbb{C}[h], \sigma, \beta h, -\beta) \qquad \text{where } \sigma(h) = h + \beta^{-1}. \tag{19}$$

In this presentation, the four-tuple $(\mathbb{C}[h], \sigma, \beta h, -\beta)$ is conformal, with $a = \frac{\beta}{1+\beta}h - (1/((1+\beta)^2))$ and Casimir element $z = yx - \frac{\beta}{1+\beta}h + (1/(\beta(1+\beta)^2))$. This presentation is similar to the presentation of $A(1-\beta, \beta, 1)$ as an iterated skew polynomial ring in [9].

3.3. *Case* C

Here we assume that $\mu_1 = \mu_2 \neq 1$ and write $\mu = \mu_1$. Thus $\alpha^2 + 4\beta = 0$, $\mu = 2/\alpha$, $\alpha \neq 2$, and $\mu^2 = -1/\beta$.

Let σ be the $\mathbb{C}$-automorphism of $\mathbb{C}[t]$ such that $\sigma(t) = \mu^{-1}t$, let $p = \mu^{-1}$, and let $c = -\mu^{-1}(t + (\mu^2\gamma/(1-\mu)))$. This time the defining equations for $R(B, \sigma, c, p)$ are

$$\mu yt = ty, \tag{20a}$$

$$xt = \mu tx, \tag{20b}$$

$$xy - \mu yx = t + \frac{\mu^2\gamma}{1-\mu}, \tag{20c}$$

the same as in Case A with $\mu_1 = \mu_2 = \mu$. As before, $R(B, \sigma, c, p) = A(\alpha, \beta, \gamma)$.

In this case $(\mathbb{C}[t], \sigma, c, p)$ is not conformal because, for $a \in \mathbb{C}[t]$, $\sigma(a) - pa$ has zero coefficient of t. By Proposition 2.1, $A(\alpha, \beta, \gamma) \simeq \mathbb{C}[t, w](\sigma, w)$, where $\sigma(t) = \mu^{-1}t$ and $\sigma(w) = \mu^{-1}(w - \mu^{-1}t - (\mu^2\gamma/(1-\mu)))$. To see that σ is conjugate to a triangular automorphism of type (c), we follow the method outlined in the proof of [30, Proposition 1b] and let $h = w - (\mu^2\gamma/((1-\mu)^2))$. Then $\mathbb{C}[t, w] = \mathbb{C}[t, h]$, and

$$A(\alpha, \beta, \gamma) \simeq \mathbb{C}[t, h](\sigma, w), \qquad \sigma(t) = \mu^{-1}t, \sigma(h) = \mu^{-1}(h - \mu^{-1}t). \tag{21}$$

3.4. *Case* D

The final case is where $\mu_1 = \mu_2 = 1$, that is, $\alpha = 2$ and $\beta = -1$. Here let σ be the $\mathbb{C}$-automorphism of $\mathbb{C}[t]$ such that $\sigma(t) = t - \gamma$, let $p = 1$, and let $c = -t$. Then $R(B, \sigma, c, p)$ is generated by x, y, and t subject to the relations

$$yt = ty - \gamma y, \tag{22a}$$

$$xt = tx + \gamma x, \tag{22b}$$

$$xy - yx = t. \tag{22c}$$

As in Case A, it is a routine matter to check that $R(B, \sigma, c, p) = A(\alpha, \beta, \gamma)$. Note that, when $\gamma = 1$, these three equations coincide with those obtained by setting $\beta = -1$ and $t = h$ in (18a), (18b), and (18c).

Suppose that $\gamma \neq 0$. Without loss of generality, $\gamma = 1$. As is observed in [8], $A(2, -1, 1) \simeq U(\mathfrak{sl}_2(\mathbb{C}))$. Here (B, σ, c, p) is conformal with $a = \frac{1}{2}(t^2 + t)$ and Casimir element $z = yx - \frac{1}{2}(t^2 - t)$. By Proposition 2.1,

$$A(\alpha, \beta, \gamma) \simeq k[t, z](\sigma, z + \sigma(a)), \qquad \sigma(t) = t - 1, \sigma(z) = z. \tag{23}$$

Thus σ is triangular of type (b) with $\lambda = 1$.

Now suppose that $\gamma = 0$. Then $(\mathbb{C}[t], \sigma, c, p)$ is not conformal because $\sigma(a) - pa = 0$ for all $a \in \mathbb{C}[t]$. By Proposition 2.1,

$$A(\alpha, \beta, \gamma) \simeq k[t, w](\sigma, w), \qquad \sigma(t) = t, \sigma(w) = w - t. \tag{24}$$

This time σ is triangular of type (c) with $\lambda = \mu = 1$. In this case $A(\alpha, \beta, \gamma)$ is isomorphic to the enveloping algebra of the Heisenberg Lie algebra.

THEOREM 3.1. *Let R be a $\mathbb{C}$-algebra. Then R is isomorphic to a down–up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0$ if and only if R is isomorphic to an ambiskew polynomial ring $R(\mathbb{C}[t], \sigma, c, p)$ for some $\mathbb{C}$-automorphism σ of $\mathbb{C}[t]$, some $p \in \mathbb{C}^*$, and some monic $c \in \mathbb{C}[t]$ with* $\deg c = 1$.

Proof. By Cases A–D, if R is isomorphic to a down–up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0$ then R is isomorphic to an ambiskew polynomial ring of the form stated.

For the converse, suppose that R is isomorphic to an ambiskew polynomial ring $R(\mathbb{C}[t], \sigma, c, p)$ as stated. Note that any $\mathbb{C}$-automorphism σ of $\mathbb{C}[t]$ is conjugate to one with $\sigma(t) = \lambda t$ for some $\lambda \in \mathbb{C}^*$ or to the automorphism σ such that $\sigma(t) = t - 1$. Thus we may assume that σ has one of these forms. In the latter case, $\sigma(t + \psi) = (t + \psi) - 1$ for all $\psi \in \mathbb{C}$ so, by changing generators, we may assume that $c = t$, in which case R is isomorphic to a down–up algebra from either Case D, if $p = 1$, or Case B, with $\gamma \neq 0$, as given in (19). So we may assume that $\sigma(t) = \lambda t$ for some $\lambda \in \mathbb{C}^*$. If $p \neq \lambda$ and neither p nor λ is 1 then $R(\mathbb{C}[t], \sigma, c, p)$ is isomorphic to a down–up algebra from Case A. If $p = 1 \neq \lambda$ then $R(\mathbb{C}[t], \sigma, c, p)$ is as in Case B, with $\beta = -\lambda$. If $\lambda = 1 \neq p$ then, possibly after applying a translation to t, $R(\mathbb{C}[t], \sigma, c, p)$ is in Case B, with $\gamma = 0$, as given in (16). If $p = \lambda \neq 1$, $R(\mathbb{C}[t], \sigma, c, p)$ is of the form in Case C. Finally, if $p = \lambda = 1$ then $R(\mathbb{C}[t], \sigma, c, p)$ is isomorphic to a down–up algebra from Case D, with $\gamma = 0$. ∎

Remark 3.2. The condition that $\deg c = 1$ can be interpreted as saying that the generator t is redundant so that the ambiskew polynomial ring is generated by x and y.

EXAMPLE 3.3. Suppose that $A(\alpha, \beta, \gamma)$ is (q, r)-differential in the sense of Section 2.1 and that $q \neq 0$. Then (10) becomes

$$q^3 X^2 - q(q+1)X + 1 = 0,$$

which has roots q^{-1} and q^{-2}. Thus either $A(\alpha, \beta, \gamma)$ is in Case A or $q = -1$, $\beta = 1$, and $\alpha = 0$, which is in Case B, or $q = 1$, $\beta = -1$ and $\alpha = 2$, which is Case D. In Case A, taking $\mu_1 = q^{-2}$ and $\mu_2 = q^{-1}$, $yt = qty$, $tx = qxt$ and $q^2xy - yx = q^2t + \frac{r}{q-1}$ and there is a normal Casimir element

$$z = yx - \frac{q^2}{q-1}t - \frac{r}{(q-1)(q^2-1)}. \tag{25}$$

In Case B, the same three relations hold and the existence of a Casimir element depends upon whether $r = 0$.

4. PRIMITIVITY

We shall only need to apply the primitivity criterion from [18] in cases where the automorphism is triangular of one of the three types listed in (11a), (11b) and (11c). The following may be extracted from [18, Proposition 7.8 and Remark 7.9(ii)].

PROPOSITION 4.1. *Let T be a generalized Weyl algebra $\mathbb{C}[t, z](\sigma, a)$ where σ is of type* (*a*), (*b*), *or* (*c*), *and $0 \neq a \in \mathbb{C}[t, z]$. Then T is primitive if and only if σ has type* (*a*) *with the subgroup $\langle\lambda, \mu\rangle$ of $\mathbb{C}^*$ having rank 2 or σ is of type* (*b*) *with λ not a root of unity or σ is of type* (*c*) *with μ not a root of unity.*

Here the term rank refers to the rank of $\langle\lambda, \mu\rangle$ as an abelian group, in the sense of [27]. From the presentations of $A(\alpha, \beta, \gamma)$ as generalized Weyl algebras in (13), (15), (17), (21), (23), and (24), we obtain the following.

THEOREM 4.2. *Suppose that $\beta \neq 0$ and let H be the subgroup of $\mathbb{C}^*$ generated by the roots of the equation* (10). *The down–up algebra $A(\alpha, \beta, \gamma)$ is primitive if and only if one of the following holds*:

(i) $\operatorname{rank} H = 2$.

(ii) $\gamma \neq 0$, $\alpha + \beta = 1$, *and β is not a root of unity.*

(iii) $\alpha^2 + 4\beta = 0$ *and β is not a root of unity.*

This result has been obtained independently by Kirkman and Kuzmanovich [23].

Remark 4.3. When $\alpha \neq 0$, there is no correlation between the ranks of H and $\langle\beta, \alpha\rangle$, as can be seen from the cases $\beta = 4, \alpha = \frac{15}{2}$, where $H = \langle 2, -1/8\rangle$ has rank 1 and $A(\alpha, \beta, \gamma)$ is not primitive, and $\alpha = 4, \beta = 2$, where $H = \langle 1 + \sqrt{6}/2, 1 - \sqrt{6}/2\rangle = \langle 1 + \sqrt{6}/2, -2\rangle$ has rank 2 and $A(\alpha, \beta, \gamma)$ is primitive.

EXAMPLE 4.4. When $A(\alpha, \beta, \gamma)$ is (q, r)-differential and $q \neq 0$ it follows from Example 3.3 that $A(\alpha, \beta, \gamma)$ is in Case A with $\operatorname{rank} H < 2$ or $\beta = 1$ and $\alpha = 0$ or $\beta = -1$ and $\alpha = 2$. Consequently $A(\alpha, \beta, \gamma)$ is not primitive.

5. FINITE-DIMENSIONAL MODULES

5.1. *Verma Modules and Finite-Dimensional Simple Modules*

Let $R = R(B, \sigma, c, p)$ be an ambiskew polynomial ring. For a maximal ideal M of B, the *Verma module* $V(M)$ is the right R-module $R/(xR + MR)$. The properties of $V(M)$ are established, in the conformal case, in [19, Proposition 2.3] and the details are easily modified to apply in general. The module $V(M)$ has a basis $\{b_i : i \geq 0\}$, where $b_i = y^i + xR + MR$. By (6b), the action of x and y on the basis elements is given by

$$b_i y = b_{i+1}, \qquad b_0 x = 0, \qquad b_i x = \lambda_{i-1} b_{i-1} \qquad \text{if } i > 0,$$

where $\lambda_{i-1} \in K$ is such that $c_i - \lambda_{i-1} \in M$, with c_i defined as in (5). When R is a down–up algebra $A(\alpha, \beta, \gamma)$, it can be checked that the sequence $\{\lambda_i\}_{\{i \geq 0\}}$ satisfies the recurrence relation

$$\lambda_i = \alpha\lambda_{i-1} + \beta\lambda_{i-2} + \gamma \qquad (i > 0),$$

where $\lambda_{-1} = 0$. There is an anti-automorphism of $A(\alpha, \beta, \gamma)$ interchanging d and u [8]. When this is used to convert a right module to a left module, the above Verma module $V(M)$ becomes the left module $V(\lambda_0)$ considered in [8]. As c has degree 1 in t, every left module $V(\lambda_0)$ occurs in this way on taking $M = (c - \lambda_0)\mathbb{C}[t]$.

Notation. If $c_d \notin M$ for all $d \geq 1$ then $V(M)$ is simple; otherwise there is a d-dimensional simple factor $L(M) := R/(MR + xR + y^d R)$, where $d \geq 1$ is minimal with $c_d \in M$. When $L(M)$ exists, we shall denote the maximal ideal $\operatorname{ann} L(M)$ by $Q(M)$.

The finite-dimensional simple modules over an ambiskew polynomial ring $R = R(B, \sigma, c, p)$ of the form considered here are given by [21, Theorem 3.1], which generalized earlier results from [16, 19].

PROPOSITION 5.1. *Let $R = R(B, \sigma, c, p)$ be an ambiskew polynomial ring and suppose that B has no periodic maximal ideals under the action of σ.*

(i) *All finite-dimensional simple modules are of the form $L(M)$.*

(ii) *All finite-dimensional R-modules are semisimple if and only if*

$$c_d \in M \qquad \Rightarrow \qquad (c_e \notin M \ (\textit{when } e \neq d) \textit{ and } M^2 + c_d B = M). \qquad (26)$$

Proof. (i) is immediate from [21, Theorem 3.1] and (ii) is [21, Theorem 3.8]. ∎

Remark 5.2. When R is a down–up algebra, the elements c_d always have degree ≤ 1 so, unless $c_d = 0$, if $c_d \in M$ then $M = c_d B = M^2 + c_d B$. Hence the criterion (26) becomes

$$c_d \in M \qquad \Rightarrow \qquad (c_e \notin M \text{ (when } e \neq d) \text{ and } c_d \neq 0). \qquad (27)$$

5.2. *Translations*

Here we consider the down–up algebras of the form $R(\mathbb{C}[t], \sigma, c, p)$, where $\sigma(t) = t + \tau$ for some $\tau \in \mathbb{C}^*$. These are Cases B and D with $\gamma \neq 0$. Without loss of generality, $\gamma = 1$. There are no periodic maximal ideals of $B[w]$ under the action of σ so Proposition 5.1(ii) is applicable. In Case D, $A(\alpha, \beta, \gamma) \simeq U(\mathfrak{sl}_2(\mathbb{C}))$ and the representation theory is well known. There is a unique d-dimensional simple module of each dimension $d \geq 1$ and all finite-dimensional modules are semisimple. In the context of this study, these are consequences of the calculation $c_d = -dt + \frac{1}{2}d(d-1)$. The next proposition deals with Case B.

PROPOSITION 5.3. *Let $R = A(1-\beta, \beta, 1)$ where $\beta \neq -1$.*

(i) *For each $d \geq 1$, R has, up to isomorphism, at most one d-dimensional simple module.*

(ii) *If β is not a root of unity the following are equivalent*:

(a) *For each $d \geq 1$, R has, up to isomorphism, a unique d-dimensional simple module.*

(b) *All finite-dimensional R-modules are semisimple.*

(c) *There are no positive integers e and d such that $d \neq e$ and*

$$d((-\beta)^e - 1) = e((-\beta)^d - 1). \tag{28}$$

(iii) *If β is a root of unity then R has no d-dimensional simple module when $(-\beta)^d = 1$ and the following are equivalent*:

(a) *For each $d \geq 1$ such that $(-\beta)^d \neq 1$, R has, up to isomorphism, exactly one d-dimensional simple module.*

(b) *All finite-dimensional R-modules are semisimple.*

(c) *There are no positive integers e and d such that $d \neq e$, $(-\beta)^d \neq 1$, $(-\beta)^e \neq 1$, and* (28) *holds.*

Proof. Let $q = -\beta$. In the presentation in (19), $c_d = \beta[d]_q h + (d - [d]_q)/(1+\beta)$ which generates a maximal ideal of $\mathbb{C}[h]$ unless $(-\beta)^d = 1$, in which case c_d is a unit. As (28) is the condition for $c_d\mathbb{C}[t] = c_e\mathbb{C}[t]$, the result follows from Proposition 5.1, taking account of Remark 5.2. ∎

Remark 5.4. The equivalence of (b) and (c) in Proposition 5.3 (ii) and (iii) has also been obtained by Carvalho and Musson [9, Proposition 5.5], with an equivalent equation in place of (28). Moreover they have shown that, in (iii), the equivalent conditions (a,b,c) all hold.

In Proposition 5.3, the set of values of β for which non-semisimple finite-dimensional $A(\alpha, \beta, \gamma)$-modules exist is countable and includes 2, where (28) holds with $d = 3$ and $e = 1$.

5.3. *Localization*

We now consider a down–up algebra R of the form $R(\mathbb{C}[t], \sigma, c, p)$ where $\sigma(t) = \lambda t$ for some $\lambda \in \mathbb{C}^*$ which is not a root of unity. These occur in Cases A, B (with $\gamma = 0$), and C. In this situation, $t\mathbb{C}[t]$ is a periodic maximal ideal of $\mathbb{C}[t]$, the localization $\mathbb{C}[t^{\pm 1}]$ has no periodic maximal ideals, and t is a normal element of $R(\mathbb{C}[t], \sigma, c, p)$. In each case the factor R/tR is isomorphic to either a quantized Weyl algebra A_1^q or a coordinate ring $C(q)$ of the quantum plane for some $q \in \mathbb{C}\backslash\{0, 1\}$ and has a factor isomorphic to either $\mathbb{C}[x]$ or $\mathbb{C}[x^{\pm 1}]$. Therefore there are finite-dimensional R-modules which are not semisimple. By [12, Lemmas 1.3 and 1.4], the localization $R_{\{t^i:i\geq 1\}}$, which will be denoted S, is isomorphic to $R(\mathbb{C}[t^{\pm 1}], \sigma, c, p)$. It is more appropriate to ask whether all finite-dimensional S-modules are semisimple. A prototype for this situation, where the answer is known to be positive, is the algebra of Woronowicz [19, 31, 35].

The results which follow deal with finite-dimensional simple S-modules. It is a consequence of [21, Theorem 3.1] that every finite-dimensional simple R-module is either annihilated by tR, in which case it is covered in Section 2.6, or is the restriction to R of a finite-dimensional S-module. As $\mathbb{C}[t^{\pm 1}]$ is σ-simple when $\sigma(t) = \lambda t$ and λ is not a root of unity, Proposition 5.1 applies to the analysis of the finite-dimensional simple S-modules. We begin with Case A, with rank $H \geq 1$, where $\lambda = \mu_2^{-1}$.

PROPOSITION 5.5. *Let R be a down–up algebra in Case* A *and let H be the subgroup of $\mathbb{C}^*$ generated by the roots of* (10). *Suppose that* rank $H \geq 1$ *and order the roots of* (10) *so that μ_2 is not a root of unity. Let $\tau = \mu_1/\mu_2$.*

(i) *For each $d \geq 1$, S has, up to isomorphism, at most one d-dimensional simple module.*

(ii) *If $\gamma = 0$ and q is not a root of unity then S has no finite-dimensional simple modules.*

(iii) *If $\gamma = 0$ and q is a primitive* nth *root of unity then the finite-dimensional simple S-modules are n-dimensional and of the form $L(M)$, where M is an arbitrary maximal ideal of $\mathbb{C}[t^{\pm 1}]$. In this case, S has finite-dimensional modules which are not semisimple.*

(iv) *If $\gamma \neq 0$ and τ is not a root of unity the following are equivalent*:

(a) *For each $d \geq 1$, S has, up to isomorphism, a unique d-dimensional simple module.*

(b) *All finite-dimensional S-modules are semisimple.*

(c) *There are no positive integers e and d such that $d \neq e$ and*

$$[e]_{\mu_1}[d]_\tau = [e]_\tau[d]_{\mu_1}. \tag{29}$$

(v) *If $\gamma \neq 0$ and τ is a root of unity then S has no d-dimensional simple module when $\tau^d = 1$ and the following are equivalent*:

(a) *For each $d \geq 1$ such that $\tau^d \neq 1$, S has, up to isomorphism, exactly one d-dimensional simple module.*

(b) *All finite-dimensional S-modules are semisimple.*

(c) *There are no positive integers e and d such that $d \neq e$, $\tau^d \neq 1$, $\tau^e \neq 1$, and* (29) *holds.*

Proof. Let $\mu = \mu_1^{-1}$. As we are in Case A, $\tau \neq 1$, $\mu \neq 1$, $\mu_1 \neq \tau$, and, as rank $H \geq 1$, μ and τ cannot both be roots of unity. Here $c_d = -\mu^d[d]_\tau t + \eta[d]_\mu$, where $\eta = \mu_2\gamma/(\mu_2 - 1)$. Thus c_d generates a maximal ideal of $\mathbb{C}[h]$ unless $\tau^d = 1$, in which case $c_d = 0$ if $\gamma = 0$ and is a unit if $\gamma \neq 0$. A routine calculation shows that, when $\gamma \neq 0$, $c_d\mathbb{C}[t] = c_e\mathbb{C}[t]$ if and only if (29) holds and the result follows from Proposition 5.1. ∎

Remark 5.6. For given d, e, and τ there are finitely many values of μ_1 for which (29) holds though not all solutions satisfy the conditions in Case A. For example, if $e = 1$ and $d = 2$, $[e]_{\mu_1}[d]_\tau = [e]_\tau[d]_{\mu_1}$ only if $\mu_1 = \tau$ which does not occur in Case A. For an example which does occur in Case A, take $\mu_1 = -3$ and $\mu_2 = \frac{-3}{2}$, so that $\tau = 2$ and $[3]_{\mu_1}[1]_\tau = [3]_\tau[1]_{\mu_1} = 7$.

EXAMPLE 5.7. Suppose that $A(\alpha, \beta, \gamma)$ is (q, r)-differential in the sense of Section 2.1 and that, as in the motivating examples, $q \neq 0$, q is not a root of unity and $r \neq 0$. As we observed in Example 3.3, $A(\alpha, \beta, \gamma)$ is in Case A with $\mu_1 = q^{-2}$ and $\mu_2 = q^{-1}$. In Proposition 5.5, $\tau = q^{-1}$ and (29) reduces to $(q^{-2e} - 1)(q^{-d} - 1) = (q^{-e} - 1)(q^{-2d} - 1)$ and hence to $q^{-e} + 1 = q^{-d} + 1$. There are no solutions with $d \neq e$ so all finite-dimensional S-modules are semisimple.

Remark 5.8. In Case A with $\gamma \neq 0$, we would be very interested to know whether, whenever rank $H = 2$, the Verma module $V(M)$ has finite length for all maximal ideals of $\mathbb{C}[t]$. This would fail to be the case if $[d]_\tau/[d]_{\mu_1}$ takes the same value infinitely often.

Next, we consider Case B, with $\gamma = 0$ and β not a root of unity, where $\sigma(t) = -\beta t$. Thus $\lambda = -\beta$.

PROPOSITION 5.9. *Let $R = A(1 - \beta, \beta, 0)$, where β is not a root of unity, and let $S = R_{\{t^i\}}$. Then S has no non-zero finite-dimensional modules.*

Proof. As $c_d = -[d]_{-\beta}t$ is a unit for all d, this is immediate from Proposition 5.1(i). ∎

Finally, we consider Case C, where $\lambda = \mu^{-1}$, and assume that μ is not a root of unity.

PROPOSITION 5.10. *Let* $R = A(\alpha, -\alpha^2/4, \gamma)$, *where* $\mu := 2/\alpha$ *is not a root of unity and let* $S = R_{\{t^i\}}$.

(i) *For each* $d \geq 1$, S *has, up to isomorphism, at most one* d*-dimensional simple module.*

(ii) *If* $\gamma = 0$ *then* S *has no finite-dimensional simple modules.*

(iii) *If* $\gamma \neq 0$ *the following are equivalent*:

(a) *For each* $d \geq 1$, S *has, up to isomorphism, a unique* d*-dimensional simple module.*

(b) *All finite-dimensional* S*-modules are semisimple.*

(c) *There are no positive integers* e *and* d *such that* $d \neq e$ *and*

$$d[e]_\mu = e[d]_\mu. \tag{30}$$

Proof. This follows from Proposition 5.1 and the calculation that $c_d = \mu^{-d}(-dt - [d]_\mu(\mu^2\gamma/(1-\mu)))$. ∎

EXAMPLE 5.11. In Proposition 5.10, the set of values of μ for which non-semisimple finite-dimensional S-modules exist is countable and includes -2, for which (30) holds for $d = 3$ and $e = 1$.

Remark 5.12. We shall not be addressing homological questions or computing Krull dimension. However, we point out that the presentations of $A(\alpha, \beta, \gamma)$ as generalized Weyl algebras in (13), (15), (17), (21), (23), and (24) make the results of [4, 6] applicable to the global and Krull dimensions of $A(\alpha, \beta, \gamma)$ and, when $A(\alpha, \beta, \gamma) = R(\mathbb{C}[t], \sigma, c, p)$ with $\sigma(t) = \lambda t$, the localization of $A(\alpha, \beta, \gamma)$ at the powers of t. In Case A, the solution to the problem in Remark 5.8 will be needed to compute the Krull dimension of the localization. In Case B, with $\gamma \neq 0$, a similar problem arises in the computation of the Krull dimension of $A(\alpha, \beta, \gamma)$ using [6, Theorem 1.1], where it is necessary to know whether, for given d and with β not a root of unity, (28) can hold for infinitely many different values of e. Taking limits as $e \to \infty$, it can be seen that this cannot occur.

6. HEIGHT ONE PRIME IDEALS

In many cases it is possible to apply the results of [17], or mild generalizations of them, to determine the height one prime ideals of $A(\alpha, \beta, \gamma)$ and, as the prime factors turn out to be familiar algebras, to derive the prime ideal structure of $A(\alpha, \beta, \gamma)$. We shall consider instances of the four cases from Section 3 where the automorphism σ has infinite order. In each case R will denote $R(\mathbb{C}[t], \sigma, c, p)$ as specified in Section 3. Whenever σ

has the form $\sigma(t) = \lambda t$, for some $\lambda \in \mathbb{C}^*$, we shall denote by S the localization $R(\mathbb{C}[t^{\pm 1}], \sigma, c, p)$. When λ is not a root of unity, $\mathbb{C}[t^{\pm 1}]$ is σ-simple so [17, 2.17] is applicable to determine the height one primes of S. This is done for ambiskew polynomial rings $R(\mathbb{C}[t^{\pm 1}], \sigma, c, p)$, with arbitrary c, in [17, Example 2.21]. Standard localization theory, e.g. [29, 2.1.16], can then be applied to determine the height one primes of R.

6.1. *Case* A

Here we consider Case A, assuming that $\operatorname{rank} H \geq 1$, where, as before, H is the subgroup of $\mathbb{C}^*$ generated by the roots μ_1 and μ_2 of (10), and that μ_2 is not a root of unity. The results depend upon whether H has rank 2 and whether $\gamma = 0$.

PROPOSITION 6.1. *Let R be a down–up algebra in Case A and suppose that* $\operatorname{rank} H = 2$.

(i) *Suppose that $\gamma = 0$.*

(a) *The height one primes of R are tR and zR.*

(b) *The height two primes of R are $zR + xR = tR + xR$ and $zR + yR = tR + yR$ and the maximal ideals are of the form $zR + (x - \xi)R + (y - \psi)R$, where $\xi, \psi \in \mathbb{C}$ and $\xi\psi = 0$.*

(ii) *Suppose that $\gamma \neq 0$.*

(a) *The height one primes of R are tR, zR, and the maximal ideals of the form $Q(c_d\mathbb{C}[t])$.*

(b) *There is a unique height two prime of R, namely $zR + tR$, and the height three prime ideals are maximal ideals of the form $zR + tR + (x - \xi)R$, where $\xi \in \mathbb{C}^*$.*

Proof. The description of the height one primes follows from [17, Example 2.21] and standard localization theory. In (i), R/tR is isomorphic to the coordinate ring $C(\mu_1)$ of the quantum plane and, by the symmetry discussed in Section 3.1, $R/zR \simeq C(\mu_2)$. In (ii), R/tR and R/zR are isomorphic to the quantized Weyl algebras $A_1^{\mu_1}$ and $A_1^{\mu_2}$, respectively. Parts (i)(b) and (ii)(b) then follow from Section 2.6. ∎

PROPOSITION 6.2. *Let R be a down–up algebra in Case* A *and suppose that* $\operatorname{rank} H = 1$. *Let n be the minimal positive integer such that $\mu_1^n \in \langle \mu_2 \rangle$ and let i be the integer such that $\mu_1^n = \mu_2^{-i}$.*

(i) *$t^i z^n$ is central in S and, for all $\tau \in \mathbb{C}^*$, $t^i z^n - \tau$ generates a height one prime of S.*

(ii) *If $i < 0$ then $(z^n - \tau t^{-i})R$ is a height one prime of R and if $i \geq 0$ then $(t^i z^n - \tau)R$ is a height one prime of R.*

(iii) *Together with* zR *and* tR, *the primes in* (ii) *are all the height one primes of* R.

Proof. This follows from [17, Example 2.21] and standard localization theory. ∎

Remark 6.3. In Proposition 6.2, R/zR and R/tR are isomorphic to either $C(\mu_1)$ and $C(\mu_2)$ or $A_1^{\mu_1}$ and $A_1^{\mu_2}$, depending on whether $\gamma = 0$. In the case $i \geq 0$, each prime $(t^i z^n - \tau)R$ either is maximal or, in the notation of Section 5.1, is contained in the maximal ideal $Q(c_d\mathbb{C}[t])$ for some d. If $i < 0$ then each prime $(z^n - \tau t^{-i})R$ is contained in the height two prime $tR + zR$ and, for some values of τ, in $Q(c_d\mathbb{C}[t])$ for some d. When $i \geq 0$ the image of t is inverted in the factor, so $R/(t^i z^n - \tau)R$ has a unique minimal non-zero ideal by [17, Theorem 4.7]. The complete classification of the spectrum will depend on whether μ_1 is a root of unity.

EXAMPLE 6.4. The (q,r)-differential down–up algebras considered in Example 5.7, where $q \neq 0$, q is not a root of unity, and $r \neq 0$, are covered by Proposition 6.2. Here $\mu_1 = q^{-2}$ and $\mu_2 = q^{-1}$ so $n = 1$ and $i = -2$. The height one primes are tR and, for $\tau \in \mathbb{C}$, $(z - \tau t^2)R$. The height two primes are $zR + tR$ and, for each $d \geq 1$, the maximal ideal $Q(c_d\mathbb{C}[t])$. From the expression for z in (25), we see that, for $\tau \in \mathbb{C}$, $R/(z - \tau t^2)R$ is isomorphic to the generalized Weyl algebra $\mathbb{C}[t](\sigma, a)$, where $\sigma(t) = q^{-1}t$ and $a = \tau t^2 + (q^2/(1-q))t + r/(q-1)(q^2-1)$. The simplicity criterion [18, 6.1] can be used to check that the localization $\mathbb{C}[t^{\pm 1}](\sigma, a)$ is simple for all but countably many values of τ. For the exceptional values, $(z - \tau t^2)R$ is contained in $Q(c_d\mathbb{C}[t])$ for some unique d as well as in the height two prime, $zR + tR$. For the other values, which include 0, $zR + tR$ is the unique height two prime containing $(z - \tau t^2)R$. The factor $R/(zR + tR)$ is isomorphic to $\mathbb{C}[x^{\pm 1}]$.

6.2. *Case* B, $\gamma = 0$

We now consider Case B, with $\gamma = 0$ and β not a root of unity. In this case $\lambda = -\beta$.

PROPOSITION 6.5. *Let* $R = A(1-\beta, \beta, 0)$ *where* β *is not a root of unity. Then the height one primes of* R *are* tR *and those of the form* $(z - \tau)R$, $\tau \in \mathbb{C}$.

Proof. As $p = 1$, this follows from the discussion in [17, Example 2.21] and standard localization theory. ∎

Remark 6.6. In Proposition 6.5, the factor R/tR is a commutative polynomial ring in two variables while, by (16), the factor $R/(z-\tau)R$ is isomorphic to the quantized Weyl algebra $A_1^{-\beta}$ if $\tau \neq 0$ and to $C(-\beta)$ if $\tau = 0$.

6.3. *Cases* B, D, $\gamma \neq 0$

In [17], only the conformal situation is considered so when $\gamma \neq 0$, it will be convenient to discuss Case B in its alternative form (19) which, taking $\beta = -1$, also covers Case D with $\gamma \neq 0$. There is no loss of generality in assuming that $\gamma = 1$. Thus $A(\alpha, \beta, \gamma) = R(\mathbb{C}[h], \sigma, h, -\beta^{-1})$, where $\sigma(h) = h + \beta^{-1}$. There is a normal Casimir element z which is central if and only if $\beta = -1$. Note that $\mathbb{C}[h]$ is σ-simple.

PROPOSITION 6.7. *Let $R = A(1 - \beta, \beta, \gamma)$, where $\gamma \neq 0$.*

(i) *If β is a root of unity then the height one primes are zR and those of the form $(z^n - \tau)R$, $\tau \in \mathbb{C}^*$, where n is such that $-\beta$ is a primitive* nth *root of unity.*

(ii) *If β is not a root of unity then the height one primes, which are all maximal, are zR and the ideals $Q(c_d\mathbb{C}[t])$, $d \geq 1$.*

Proof. The classification of the height one primes follows from [17, 2.17 and 2.18] using standard localization theory. In (ii), the maximality of zR can be deduced easily from [18, Theorem 6.1]. ∎

Remark 6.8. In Proposition 6.7(i), all but countably many of the ideals $(z^n - \tau)R$ are maximal. Each exception is contained in $Q(c_d\mathbb{C}[t])$ for some d.

6.4. *Case* D, $\gamma = 0$

PROPOSITION 6.9. *Let* $R = A(2, 1, 0)$. *The height one primes of R have the form $(t - \tau)R$, $\tau \in \mathbb{C}$. The factor R/tR is a polynomial ring in two variables and if $\tau \neq 0$, $R/(t - \tau)R \simeq A_1(\mathbb{C})$.*

Proof. Here t is central and the localization of R at $\mathbb{C}[t]\backslash\{0\}$ is simple, being isomorphic to the Weyl algebra $A_1(\mathbb{C}(t))$. The result follows easily. ∎

6.5. *Case* C

Here we consider Case C with μ not a root of unity. As $(\mathbb{C}[t], \sigma, c, p)$ is not conformal, some adjustment to the results of [17] is needed. As $\mathbb{C}[t^{\pm 1}]$ is σ-simple, the following results of Wells [33] apply.

PROPOSITION 6.10. *If* char $K = 0$, *B is σ-simple and (B, σ, c, p) is not conformal then $B[w]$ is σ-simple, where $\sigma(w) = pw + \sigma(c)$.*

PROPOSITION 6.11. *If $B[w]$ is σ-simple and $c_d \neq 0$ for all $d \geq 1$ then the localizations of $R(B, \sigma, c, p)$ at $\{y^i\}_{\{i\geq 1\}}$ and $\{x^i\}_{\{i\geq 1\}}$ are simple.*

In the proof of Proposition 5.10, we saw that $c_d = \mu^{-d}(-dt - [d]_\mu(\mu^2\gamma/(1-\mu)) \neq 0$ for all d so the localizations S_y and S_x are both simple. As $\mathbb{C}[t^{\pm 1}]$ has Krull dimension 1, the following result is obtained by adapting the last two paragraphs of the proof of [17, Theorem 2.12]. Throughout one replaces $u - \rho^d\sigma(u)$ by c_d.

PROPOSITION 6.12. *Let Q be a prime ideal of S containing x^m and y^m for some $m \geq 1$. Then there exists a maximal ideal M of $\mathbb{C}[t^{\pm 1}]$ such that $c_d \in M$ for some $d \geq 1$ and $Q = Q(M)$.*

PROPOSITION 6.13. *Let $R = A(\alpha, -\alpha^2/4, \gamma)$ where $\alpha \neq 0$ and $\mu := 2/\alpha$ is not a root of unity.*

(i) *If $\gamma = 0$ then tR is the unique height one prime of R and $R/tR \simeq C(\mu)$.*

(ii) *If $\gamma \neq 0$ then the height one primes of R are tR and the ideals of the form $Q(c_d\mathbb{C}[t])$, $d \geq 1$. Here $R/tR \simeq A_1^\mu$ and each $Q(c_d\mathbb{C}[t])$ is maximal.*

Proof. It follows from Propositions 6.10, 6.11, and 6.12 that every height one prime of S has the form $Q(M)$ for a maximal ideal M of $\mathbb{C}[t^{\pm 1}]$ such that $c_d \in M$ for some $d \geq 1$. Given that $S = R_{\{t^i : i \geq 1\}}$, the result follows on the observations that, in (i), each c_d is a unit and that, in (ii), each $c_d\mathbb{C}[t]$ is maximal and $t \notin Q(c_d\mathbb{C}[t])$. ∎

7. THE CASE $\beta = 0$

7.1. *Ambiskew Polynomial Rings with Arbitrary Endomorphisms*

In the generalization of ambiskew polynomial ring to the case where σ need not be injective, one skew polynomial extension is written with coefficients on the right and one with coefficients on the left. Thus we need to use notation which distinguishes between the two. For skew polynomial rings with their coefficients on the left, we use the standard notation $R[x; \sigma, \delta]$, and for those with their coefficients on the right, we shall use the non-standard notation $[\delta, \sigma; x]R$ or, if $\delta = 0$, $[\alpha; x]R$. The construction in Section 2.2 can now be extended to the case of an arbitrary endomorphism σ of B and an arbitrary scalar p. With B, c, and p as before, except that p may now be 0, let σ be a K-endomorphism of B. Let $S = [\sigma; x]B$ and extend σ to S by setting $\sigma(x) = px$. As before, there is a (left) τ-derivation δ of S such that $\delta(B) = 0$ and $\delta(x) = c$ and the *ambiskew polynomial ring* $R(B, \sigma, c, p)$ is defined to be $S[y; \sigma, \delta]$. Thus $yx - pxy = c$ and, for all $b \in B$, $bx = x\sigma(b)$ and $yb = \sigma(b)y$. Note that $R(B, \sigma, c, p)$ can also be written as $[\delta', \sigma; x](B[y; \sigma])$, where $\sigma(y) = py$, $\delta'(B) = 0$, and $\delta'(y) = c$.

To realise $A(\alpha, 0, \gamma)$ in the form $R(B, \sigma, c, p)$, take $B = \mathbb{C}[t]$ and let σ be the $\mathbb{C}$-endomorphism of B such that $\sigma(t) = 0$, let $p = \alpha$, and let $c = t + \gamma$.
Thus

$$tx = 0 = yt, \tag{31}$$

$$yx - \alpha xy = t + \gamma, \tag{32}$$

and, using (32) to substitute for t in (31), we obtain

$$y^2 x = \alpha yxy + \gamma y, \tag{33a}$$

$$yx^2 = \alpha xyx + \gamma x. \tag{33b}$$

These are the down–up relations (1a) and (1b) with $\beta = 0$, $y = d$, and $x = u$. Conversely, given x, y satisfying (33a) and (33b), and setting $t = yx - \alpha xy - \gamma$ leads to (31). Consequently

$$A(\alpha, 0, \gamma) = R(\mathbb{C}[t], \sigma, t + \gamma, \alpha). \tag{34}$$

THEOREM 7.1. *Let R be a $\mathbb{C}$-algebra. Then R is isomorphic to a down–up algebra $A(\alpha, \beta, \gamma)$ if and only if R is isomorphic to an ambiskew polynomial ring $R(\mathbb{C}[t], \sigma, c, p)$ for some $\mathbb{C}$-endomorphism σ of $\mathbb{C}[t]$, some $p \in \mathbb{C}$, and some $c \in \mathbb{C}[t]$ with* $\deg \sigma(t) \leq 1$ *and* $\deg c = 1$.

Proof. By Theorem 3.1 and (34), it suffices to show that if $R \simeq R(\mathbb{C}[t], \sigma, c, p)$ where $\deg c = 1$ and either $\sigma(t) \in \mathbb{C}$ or σ is bijective and $p = 0$ then R is isomorphic to a down–up algebra. If $\sigma(t) \in \mathbb{C}$ then, by a change of variables, we can assume that $\sigma(t) = 0$, in which case the result follows from (34). If σ is an automorphism of $\mathbb{C}[t]$ and $p = 0$, it suffices to consider the case where $\sigma(t) = \lambda t$, $\lambda \in \mathbb{C}^*$, and $c = t + \eta$ and the case where $\sigma(t) = t - 1$ and $c = t$. These both give rise to alternative presentations of down–up algebras of the form $A(\alpha, 0, \gamma)$. In the former case $R(\mathbb{C}[t], \sigma, t + \eta, 0) = A(\lambda, 0, \eta(1 - \lambda))$ and in the latter, $R(\mathbb{C}[t], \sigma, t, 0) = A(1, 0, -1)$. ∎

The proof that $A(\alpha, 0, \gamma)$ is not right Noetherian given in [25] extends to the general case of $R(B, \sigma, c, p)$ when σ is not injective.

PROPOSITION 7.2. *Let R be a ring, let τ be an endomorphism of R, and let δ be a τ-derivation of R.*

(i) *If τ is not injective then $[\tau; x]R$ is not right Noetherian.*

(ii) *If R is not right Noetherian then $R[x; \tau, \delta]$ is not right Noetherian.*

Proof. (i) Let $0 \neq a \in R$ be such that $\tau(a) = 0$. Let $S = [\tau; x]R$ and, for $j \geq 1$, let $I_j = \sum_{i=1}^{j} x^i aS$. Then $ax = 0$ so $aS = aR$, whence $I_j = \sum_{i=1}^{j} x^i aR$ and $x^{j+1} a \notin I_j$. Thus S is not right Noetherian.

(ii) Let $S = R[x; \tau, \delta]$. As S is free as a left R-module, if $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ is a strictly ascending chain of right ideals of R then $I_1 S \subset I_2 S \subset \cdots \subset I_n S \subset \cdots$ is a strictly ascending chain of right ideals of S. ∎

COROLLARY 7.3. *Let σ be a non-injective K-endomorphism of a commutative K-algebra B, let $c \in B$, and let $p \in K$. Then $R(B, \sigma, c, p)$ is neither right nor left Noetherian.*

Proof. By Proposition 7.2, $R(B, \sigma, c, p)$ is not right Noetherian. As we observed in Section 7.1, $R(B, \sigma, c, p)$ can be presented in the alternative form $[\delta', \sigma; x](B[y; \sigma])$ and it follows, by symmetry, that $R(B, \sigma, c, p)$ is not left Noetherian. ∎

Combining this with Theorems 3.1 and 7.1 and applying standard results on skew polynomial rings, we obtain an alternative proof of the main result of [25]. In Theorem 7.1, the following are equivalent:

(i) R is Noetherian.

(ii) $\beta \neq 0$.

(iii) σ is an automorphism and $p \neq 0$.

(iv) $\deg \sigma(t) = 1$ and $p \neq 0$.

(v) R is a domain.

7.2. *Casimir Elements*

As before, when $c = \sigma(a) - pa$ for some $a \in B$, we shall say that (B, σ, c, p) is *conformal*. If this is the case and $z := yx - \sigma(a) = p(xy - a)$ then $zb = bz$ for all $b \in B$, $yz = pzy$ and $zx = pxz$. If $p \neq 0$ then z is a normal element of $R(B, \sigma, c, p)$.

EXAMPLE 7.4. Let $R = A(\alpha, 0, \gamma) = R(\mathbb{C}[t], \sigma, t + \gamma, \alpha)$, where $\sigma(t) = 0$. The four-tuple $(\mathbb{C}[t], \sigma, t + \gamma, \alpha)$ is conformal when $\alpha \neq 0, 1$. To see this, suppose that $\alpha \neq 0, 1$ and let $a = -\frac{1}{\alpha}t + \frac{\gamma}{1-\alpha}$. Then $\sigma(a) - \alpha a = t + \gamma$ so R has a normal element

$$z = \alpha xy + t - \frac{\alpha\gamma}{1-\alpha} = yx - \frac{\gamma}{1-\alpha},$$

such that $zt = tz$, $yz = \alpha zy$, and $zx = \alpha xz$. The existence of this normal element has previously been observed in [9, 6.1]. As is stated there, the $\mathbb{C}$-algebra R/zR is generated by $X := x + zR$ and $Y := (1-\alpha)y + zR$, subject to the relation $YX = \gamma$. We shall denote this algebra by J_γ. When $\gamma \neq 0$ it is isomorphic to J_1 which is isomorphic to a well-known primitive ring [14, p. 35, Example 2]. The algebra J_γ has an interpretation as an ambiskew polynomial ring, namely $R(\mathbb{C}, \mathrm{id}, \gamma, 0)$.

The following result, which will be applied later to other rings, is applicable to the ideal structure of J_1.

PROPOSITION 7.5. *Let R be a ring, let N be a maximal right ideal of R, and let $P = \operatorname{ann}(R/N)$. Let $t \in R$ be such that $tN \subseteq P$ and let Q be any ideal of R such that $P \subset Q$. Then $t \in Q$.*

Proof. As $Q \nsubseteq N$, $N + Q = R$ so $1 = n + q$ for some $n \in N$ and some $q \in Q$. Then $t = tn + tq \in P + Q \subseteq Q$. ∎

COROLLARY 7.6. *The ring J_1 is primitive and has a unique minimal non-zero ideal $P := J_1(XY - 1)J_1$. The prime ideals of J_1 are 0, P, and the ideals of the form $P + (Y - \tau)J_1$, $\tau \in \mathbb{C}^*$.*

Proof. The right ideal $N := XJ_1$ is maximal and $\operatorname{ann}(J_1/N) = 0$. If $t = XY - 1$ then $tN = 0$ so, by Proposition 7.5, every non-zero ideal of J_γ contains P. The rest follows from the fact that J_1/P is isomorphic to the Laurent polynomial ring $\mathbb{C}[Y^{\pm 1}]$. ∎

COROLLARY 7.7. *Let $R = A(\alpha, 0, \gamma)$, where $\alpha \neq 0, 1$. The ideal zR is prime if and only if $\gamma \neq 0$. If $\gamma \neq 0$ then every ideal of R strictly containing zR contains $xy - \gamma$.*

Proof. We have observed that $R/zR \simeq J_\gamma$ and that if $\gamma \neq 0$ then J_γ is primitive, by Corollary 7.6, and hence prime. As $XYJ_0X = 0$, J_0 is not prime. The final statement is immediate from Corollary 7.6. ∎

7.3. *Skew Commutator Formulae*

The skew commutator formulae (6a) and (6b) remain valid, with $c_d := \sum_{j=0}^{d-1} p^{d-j-1}\sigma^j(c)$ as in (5), as do their consequences (7) and, in the conformal case, (8).

In $A(\alpha, 0, \gamma)$, $c = t + \gamma$, $\sigma^j(c) = \gamma$ for $j > 0$ and $c_d = \alpha^{d-1}t + [d]_\alpha\gamma$. This is valid not only in the conformal case but also when $\alpha = 1$, in which case $c_d = t + d\gamma$, and when $\alpha = 0$, in which case $c_d = \gamma$ for $d > 1$.

7.4. *Ideals of $A(\alpha, 0, \gamma)$*

Let $R = R(B, \sigma, c, p)$, let $S = [\sigma; x]B$, and let I be an ideal of B such that $\sigma(I) \subseteq I$. Then SI is an ideal of S consisting of all elements of the form $\sum_{j=0}^{n} x^j i_j$, $i_j \in I$. The factor S/SI is isomorphic to $[\sigma; x](B/I)$, where here σ is the induced endomorphism of B/I. Furthermore, the ideal SI is invariant under σ and δ, whence $SIR = RIR$ is an ideal of R consisting of finite sums of elements of the form $x^j i y^k$, $i \in I$, $j, k \geq 0$. The factor R/RIR is isomorphic to $R(B/I, \sigma, \bar{c}, p)$, where $\bar{c} = c + I$.

Now consider the ring $R = A(\alpha, 0, \gamma) = R(\mathbb{C}[t], \sigma, t + \gamma, \alpha)$, where $\sigma(t) = 0$. For each $f \in \mathbb{C}[t]$, $\sigma(ft\mathbb{C}[t]) = 0$ and so the ideal $RftR$ consists of finite sums of elements of the form $x^j ftby^k$, $b \in \mathbb{C}[t]$, $j, k \geq 0$. The factor $R/RftR$ is isomorphic to $R(\mathbb{C}[t]/ft\mathbb{C}[t], \sigma, \overline{t + \gamma}, \alpha)$.

PROPOSITION 7.8. *The ideal RtR is prime if and only if $\alpha \neq 0$ or $\gamma \neq 0$.*

Proof. The factor R/RtR is isomorphic to $R(\mathbb{C}, \mathrm{id}, \gamma, \alpha)$. Suppose that $\alpha \neq 0$. If $\gamma = 0$, then $R(\mathbb{C}, \mathrm{id}, \gamma, \alpha)$ is isomorphic to $C(\alpha)$ and if $\gamma \neq 0$, it is isomorphic to A_1^{α}. Both are domains so RtR is a prime ideal.

If $\alpha = 0$ then $R/RtR \simeq J_{\gamma}$, which, by the proof of Corollary 7.7, is prime if and only if $\gamma \neq 0$. ∎

Remark 7.9. The down–up algebra $A(0,0,0)$ is the $\mathbb{C}$-algebra R generated by d, u subject to the relations $d^2u = du^2 = 0$. Here $\sum_{j\geq 2} u^j R$ is a non-zero ideal of R annihilated on the left by d so $A(0,0,0)$ is not prime.

THEOREM 7.10. *Let $R = A(\alpha, 0, \gamma) = R(\mathbb{C}[t], \sigma, t+\gamma, \alpha)$, where $\sigma(t) = 0$. Suppose that $\alpha \neq 0$ or $\gamma \neq 0$.*

(i) *Every non-zero ideal of R contains an ideal of the form $RftR$, $0 \neq f \in \mathbb{C}[t]$.*

(ii) *R is a prime ring.*

Proof. (i) Let $S = [\sigma; x]\mathbb{C}[t]$. Note that, for $d \geq 1$, $c_d = \alpha^{d-1}t + [d]_{\alpha}\gamma$ is not a right zero-divisor in S. Let I be a non-zero ideal of R. We first show that $I \cap S \neq 0$. Let $a = \sum_{j=m}^{n} f_j y^j \in I$, where each $f_j \in S$, f_n and f_m are non-zero, and m is minimal for non-zero elements of I. Suppose that $m > 0$. By (6b),

$$ax = \sum_{j=m}^{n} (\alpha^j f_j x y^j + f_j c_j y^{j-1}) \in I.$$

As c_m is not a right zero-divisor in S, this contradicts the minimality of m. Hence $m = 0$, $f_0 \neq 0$, and $a = f_0 + \sum_{j=1}^{n} f_j y^j$. But $yt = 0$ so $0 \neq f_0 t = at \in I \cap S$.

Now let $b = \sum_{j=m}^{n} x^j f_j \in I \cap S$, where each $f_j \in \mathbb{C}[t]$, f_n and f_m are non-zero, and m is minimal for non-zero elements of $I \cap S$. If $m > 0$ then, by (6a) and as $yt = 0$,

$$ybt = \sum_{j=m}^{n} (\alpha^j x^j y f_j t + x^{j-1} c_j f_j t) = \sum_{j=m}^{n} x^{j-1} c_j f_j t \in I \cap S,$$

which, as c_m is not a zero-divisor in $\mathbb{C}[t]$, would contradict the minimality of m. Hence $m = 0$, $f_0 \neq 0$, and $b = f_0 + \sum_{j=1}^{n} x^j f_j$. But $tx = 0$ so $0 \neq tf_0 = bt \in I \cap S$. Therefore $Rtf_0R \subseteq I$ as required.

(ii) This is immediate from (i) as every non-zero ideal of R has non-zero intersection with the domain $\mathbb{C}[t]$. ∎

Remark 7.11. Theorem 7.10(ii) has been proved independently by Kirkman and Kuzmanovich [24], who have also shown that $A(\alpha, 0, \gamma)$ is not primitive.

Remark 7.12. If $\alpha \neq 0, 1$ then we have seen that R has a normal element $z = \alpha xy + t - \frac{\alpha\gamma}{1-\alpha}$. By Theorem 7.10(ii), $ft \in zR$ for some $f \in \mathbb{C}$. In fact, as $tx = 0$, $tz = t^2 - \frac{\alpha\gamma}{1-\alpha}t \in zR$.

PROPOSITION 7.13. *Let $0 \neq f \in \mathbb{C}[t]$ be monic and let P be a prime ideal of R containing $RftR$. If $RtR \nsubseteq P$ then $Rt(t-\eta)R \subseteq P$ for some $\eta \in \mathbb{C}^*$. Consequently, if $RftR$ is prime then either $f = 1$ or $f = t - \eta$ for some $\eta \in \mathbb{C}^*$.*

Proof. Suppose that $RtR \nsubseteq P$. Then $f \neq 1$ so f factorizes into factors of the form $t - \eta$, $\eta \in \mathbb{C}$. As $tx = yt = 0$, it follows from (7) that, for $g, h \in \mathbb{C}[t]$, $(RgtR)(RhtR) \subseteq Rght^2R$. Hence $Rt(t-\eta)R \subseteq P$ for some $\eta \in \mathbb{C}$. Moreover $\eta = 0$; otherwise $(RtR)^2 \subseteq P$ and $RtR \subseteq P$. For the consequence, note that, for $f_1, f_2 \in \mathbb{C}[t]$, $Rf_1R \subseteq Rf_2R$ if and only if $f_1 \in f_2\mathbb{C}[t]$. ∎

Further analysis of the prime spectrum of $A(\alpha, 0, \gamma)$ will be given after analysis of the finite-dimensional simple $A(\alpha, 0, \gamma)$-modules.

8. FINITE-DIMENSIONAL SIMPLE MODULES, $\beta = 0$

8.1. *Verma Modules*

For ambiskew polynomial rings in general, the definition and notation for Verma modules will be as in Section 5.1 where σ was an automorphism. Thus, for a maximal ideal of B, $V(M) := R/(MR + xR)$. If $c_d \notin M$ for all $d \geq 1$ then $V(M)$ is simple. If $c_d \in M$ for some $d \geq 1$ then, as in Section 5.1, for the minimal such d, $L(M)$ will denote $R/(MR + xR + y^dR)$, which is a simple d-dimensional R-module, and $Q(M)$ will denote $\operatorname{ann} L(M)$.

Now let $R = A(\alpha, 0, \gamma) = R(\mathbb{C}[t], \sigma, t + \gamma, \alpha)$ with $\sigma(t) = 0$. Note that, as $yt = 0$, $t(t-\eta)$ annihilates the Verma module $V((t-\eta)\mathbb{C}[t])$. We have seen that $c_d = \alpha^{d-1}t + [d]_\alpha\gamma$ so, if $\alpha \neq 0$, then $c_d\mathbb{C}[t]$ is a maximal ideal of $\mathbb{C}[t]$. The finite-dimensional modules of the form $L(M)$ which arise are as follows.

Case (i). If $\alpha = \gamma = 0$ then $c_1 = t$ and $c_d = 0$ if $d > 1$ so there is a one-dimensional simple module $L(t\mathbb{C}[t])$ and, for each $\eta \in \mathbb{C}^*$, there is a two-dimensional simple module $L((t-\eta)\mathbb{C}[t])$.

Case (ii). If $\alpha = 0$ and $\gamma \neq 0$ then $c_1 = t + \gamma$ and $c_d = \gamma$ if $d > 1$ so there is a one-dimensional simple module $L((t+\gamma)\mathbb{C}[t])$ and $V((t-\eta)\mathbb{C}[t])$ is simple if $\eta \neq -\gamma$.

Case (iii). If $\alpha \neq 0$ and $\gamma = 0$ then $c_d\mathbb{C}[t] = t\mathbb{C}[t]$ for all d so there is a one-dimensional simple module $L(t\mathbb{C}[t])$ and $V((t-\eta)\mathbb{C}[t])$ is simple if $\eta \neq 0$.

Case (iv). Suppose that $\alpha \neq 0$ and $\gamma \neq 0$. If $\alpha \neq 1$ then, for $e, d \geq 1$, $c_d\mathbb{C}[t] = c_e\mathbb{C}[t]$ if and only if $\alpha^d = \alpha^e$. Hence if α is not a root of unity there is a d-dimensional simple module $L(c_d\mathbb{C}[t])$ for each $d \geq 1$ and if α is a primitive nth root of unity, where $n > 1$, there is a d-dimensional simple module $L(c_d\mathbb{C}[t])$ for each d with $n \geq d \geq 1$. If $\alpha = 1$ then $c_d = t + d\gamma$ so $c_d\mathbb{C}[t] = c_e\mathbb{C}[t]$ if and only if $d = e$, whence there is a d-dimensional simple module $L(c_d\mathbb{C}[t])$ for each $d \geq 1$.

8.2. *Classification*

The proof of the following classification theorem is adapted from that of [19, Theorem 2.6].

THEOREM 8.1. *Let R be a down–up algebra of the form $A(\alpha, 0, \gamma)$. Let X be a finite-dimensional simple right R-module. Either $Xt = 0$, in which case X is a simple R/RtR-module, or $X \simeq L(M)$ for some maximal ideal M of $\mathbb{C}[t]$.*

Proof. Let $B = \mathbb{C}[t]$. Suppose that $Xy^d \neq 0$ for all $d \geq 0$. The descending chain of B-submodules $X \supseteq Xy \supseteq Xy^2 \cdots$ must terminate in a non-zero term $V = Xy^d = Xy^{d+1}$ for some $d \geq 1$. By (6b),

$$Vx = Xy^{d+1}x \subseteq Xxy^{d+1} + Xc_{d+1}y^d \subseteq Xy^{d+1} + Xy^d = V,$$

so V is an R-submodule of X, whence $X = V = Xy^d$ and, as $yt = 0$, $Xt = 0$.

We can now suppose that $Xt \neq 0$ and that there exists $e \geq 0$ such that $Xy^e = 0$. As $Xtx = 0$, there exists $w \in X\backslash\{0\}$ such that $wx = wy^d = 0$ for some positive integer d. Choose such an element w with d minimal. Let $a \in B$ and let $i \geq 1$. By (6b),

$$way^ix = \alpha^i waxy^i + wac_iy^{i-1} = wac_iy^{i-1}. \tag{35}$$

Let $M = \operatorname{ann}_B(w)$ and let I be an ideal of B strictly containing M. By (35), $wI + wIy + \cdots + wIy^{d-1}$ is an R-submodule of X and hence, by the simplicity of X,

$$X = wI + wIy + \cdots + wIy^{d-1} = wB + wBy + \cdots + wBy^{d-1}.$$

Consequently, there exist $i_1, i_2, \ldots, i_d \in I$ such that

$$w = wi_1 + wi_2y + \cdots + wi_dy^{d-1}.$$

But then $w(1 - i_1)y^{d-1} = 0 = w(1 - i_1)x$, contradicting the choice of d unless $w(1 - i_1) = 0$. Hence $1 - i_1 \in M \subset I$, whence $1 \in I$ and $I = B$. Therefore M is a maximal ideal of B. Now $wc_dx = wx\sigma(c_d) = 0$ and, by (35), $0 = wy^dx = wc_dy^{d-1}$. By the minimality of d, $c_d \in M$. If $c_f \in M$

for some f with $1 \leq f < d$ then $wy^f x = 0 = wy^f y^{d-f}$, contradicting the choice of w and d. Thus d is minimal with $c_d \in M$. Now $MR + xR + y^d R \subseteq \operatorname{ann}_R(w)$ so, by the simplicity of $L(M)$, $MR + xR + y^d R = \operatorname{ann}_R(w)$ and $X \simeq R/\operatorname{ann}_R(w) = L(M)$. ∎

The simple R-modules of the form $L(M)$, where $R = A(\alpha, 0, \gamma)$, have already been classified in Section 8.1. To complete the classification, it is necessary to classify the finite-dimensional modules annihilated by t. We present the results below where S denotes the factor R/RtR and $\bar{x}$ and $\bar{y}$ denote the images of x and y in S. The cases to be considered are as in Section 8.1.

In Case (i), where $\alpha = \gamma = 0$ and $t = yx$, $R/RtR \simeq J_0$. The finite-dimensional simple R-modules annihilated by RtR are one-dimensional and of the form $R/((y - \xi)R + (x - \psi)R)$, $\xi\psi = 0$. In Case (ii), where $\alpha = 0$, $\gamma \neq 0$, and $t = yx - \gamma$, $R/RtR \simeq J_\gamma$. The finite-dimensional simple R-modules annihilated by RtR are one-dimensional and of the form $R/((y - \xi)R + (x - \xi^{-1}\gamma)R)$, $\xi \neq 0$.

In Case (iii), $R/RtR \simeq C(\alpha)$ and, in Case (iv), $R/RtR \simeq A_1^\alpha$. In both these cases, the finite-dimensional simple R-modules annihilated by t are classified as in Section 2.6, with $q = \alpha$. In particular, when $\alpha = 1$ and $\gamma \neq 0$ there are no such modules. For $R = A(1, 0, 1)$, $c_d = t + d$ so there is a unique d-dimensional simple module $L((t + d)\mathbb{C}[t])$ for each $d \geq 1$ and these are all the finite-dimensional simple R-modules.

9. THE PRIME SPECTRUM OF $A(\alpha, 0, \gamma)$

9.1. *Annihilators of Simple Verma Modules*

To analyse the prime spectrum of $A(\alpha, 0, \gamma)$, we shall need to identify the annihilator of the Verma module $V((t - \eta)\mathbb{C}[t])$ whenever it is simple. Throughout this section, R will denote the down–up algebra $A(\alpha, 0, \gamma)$, which, by (34), is $R(\mathbb{C}[t], \sigma, t + \gamma, \alpha)$, where $\sigma(t) = 0$.

THEOREM 9.1. *Suppose that α is non-zero and not a root of unity. For $\eta \in \mathbb{C}$, let M_η be the maximal ideal $(t - \eta)\mathbb{C}[t]$. Suppose that $\eta \neq 0$, $\eta \neq \frac{\alpha\gamma}{1-\alpha}$ and that, for all $d \geq 1$, $\alpha^{d-1}\eta + [d]_\alpha\gamma \neq 0$. Let $P_\eta = Rt(t - \eta)R$. Then P_η is the annihilator of the Verma module $V(M_\eta)$.*

Proof. Let $S = \mathbb{C}[t][y; \sigma]$ and extend σ to S by setting $\sigma(y) = y$. This is not the same extension of σ as in Section 7.1 but is such that $ys = \sigma(s)y$ for all $s \in S$. Note that $\sigma(S) = \mathbb{C}[y]$ and $\ker \sigma = tS$.

The condition $\alpha^{d-1}\eta + [d]_\alpha\gamma \neq 0$ ensures that $V(M_\eta)$ is simple. Let $M = M_\eta R + xR$ so that M is maximal and $V(M_\eta) = R/M$. Then $V(M_\eta)$ has a $\mathbb{C}$-basis $\{b_i\}_{i\geq 0}$ where, for $i \geq 0$, $b_i := y^i + M$. The annihilator in $\mathbb{C}[t]$ of b_i

is $\sigma^{-i}(M_\eta)$ which is $t\mathbb{C}[t]$ if $i > 0$ and M_η if $i = 0$. Thus $t(t-\eta) \in \operatorname{ann}_{\mathbb{C}[t]} b_i$ for all i so $t(t-\eta) \in \operatorname{ann} V(M_\eta)$, whence $P_\eta \subseteq \operatorname{ann} V(M_\eta)$.

To establish the reverse inclusion, suppose that it is false and choose $b = \sum_{i=0}^n x^i f_i \in \operatorname{ann} V(M_\eta)$, with each $f_i \in S$, such that $f_n \notin t(t-\eta)S$.

Let $\psi = \eta - \frac{\alpha\gamma}{1-\alpha}$, which is non-zero by our hypothesis on η, and let $\tau = \frac{\gamma}{1-\alpha}$. For $A(\alpha, 0, \gamma)$, the formula (8) gives that, for $0 \le d \le n$,

$$y^d x^d = (\alpha^{d-1}z + \tau)(\alpha^{d-2}z + \tau)\cdots(z+\tau), \tag{36}$$

where, as in Example 7.4, $z = \alpha xy + t - \frac{\alpha\gamma}{1-\alpha}$, whence $z \equiv \psi \bmod M$. By (36), for $0 \le d \le n$,

$$y^d x^d f_d \equiv (\alpha^{d-1}\psi + \tau)(\alpha^{d-2}\psi + \tau)\cdots(\psi+\tau) f_d \bmod M. \tag{37}$$

As $yz = \alpha zy$ and $ys = \sigma(s)y$ for all $s \in S$ and as σ is idempotent, it follows from (36) that, for $j \ge 1$ and $0 \le d \le n$,

$$y^{d+j} x^d f_d = (\alpha^{d-1+j}z + \tau)(\alpha^{d-2+j}z + \tau)\cdots(\alpha^j z+\tau)\sigma(f_d) y^j. \tag{38}$$

Hence, for $j \ge 1$ and $0 \le d \le n$,

$$y^{d+j} x^d f_d \equiv (\alpha^{d-1+j}\psi + \tau)(\alpha^{d-2+j}\psi + \tau)\cdots(\alpha^j \psi+\tau)\sigma(f_d) y^j \bmod M. \tag{39}$$

Let $0 \le d, i \le n$. It follows from (39), with $j = n + i - d + 1$ that, modulo M,

$$\begin{aligned} 0 \equiv y^{n+i+1} b y^{n-i-1} &\equiv y^{n+i+1}\Big(\sum_{d=0}^n x^d f_d\Big) y^{n-i-1} \\ &\equiv \sum_{d=0}^n (\alpha^{n+i}\psi + \tau)(\alpha^{n+i-1}\psi + \tau)\cdots(\alpha^{n+i+1-d}\psi + \tau)\sigma(f_d) y^{2n-d}. \end{aligned}$$

Thus $C(\eta, \tau)V = 0$ where $C(\eta, \tau)$ is the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & \alpha^n\psi + \tau & (\alpha^n\psi+\tau)(\alpha^{n-1}\psi+\tau) & \cdots & \prod_{1\le s\le n}(\alpha^s\psi+\tau) \\ 1 & \alpha^{n+1}\psi + \tau & (\alpha^{n+1}\psi+\tau)(\alpha^{n}\psi+\tau) & \cdots & \prod_{2\le s\le n+1}(\alpha^s\psi+\tau) \\ 1 & \alpha^{n+2}\psi + \tau & (\alpha^{n+2}\psi+\tau)(\alpha^{n+1}\psi+\tau) & \cdots & \prod_{3\le s\le n+2}(\alpha^s\psi+\tau) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{2n}\psi + \tau & (\alpha^{2n}\psi+\tau)(\alpha^{2n-1}\psi+\tau) & \cdots & \prod_{n+1\le s\le 2n}(\alpha^s\psi+\tau) \end{pmatrix},$$

with entries in $\mathbb{C}$ and V is the $(n+1) \times 1$ matrix

$$(\sigma(f_0)y^{2n} \quad \sigma(f_1)y^{2n-1} \quad \cdots \quad \sigma(f_n)y^n)^T$$

with entries in $\mathbb{C}[y]$.

For $1 \leq i, j \leq n+1$ the ijth entry, $c_{i,j}$ say, of $C(\eta, \tau)$ is

$$(\alpha^{n+i-j+1}\psi + \tau)(\alpha^{n+i-j+2}\psi + \tau)\cdots(\alpha^{n+i-1}\psi + \tau).$$

By (9),

$$c_{i,j} = \sum_{k=0}^{j-1} \begin{bmatrix} j-1 \\ k \end{bmatrix}_\alpha \alpha^{(k(k-1)/2)} \tau^{j-1-k} (\alpha^{n+i-j+1}\psi)^k. \tag{40}$$

Hence

$$c_{i,j} = \sum_{k=0}^{j-1} d_{j,k} \alpha^{k(i-1)}, \tag{41}$$

where $d_{j,k} = \begin{bmatrix} j-1 \\ k \end{bmatrix}_\alpha \alpha^{(k(k-1)/2)} \tau^{j-1-k} (\alpha^{n-j+2}\psi)^k$. In particular,

$$d_{j,j-1} = \alpha^{((j-1)(j-2)/2)} (\alpha^{n-j+2}\psi)^{j-1} \neq 0.$$

It follows that $C(\eta, \tau)$ has the same column space as the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \alpha & \alpha^2 & \ldots & \alpha^n \\ 1 & \alpha^2 & \alpha^4 & \ldots & \alpha^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \ldots & \alpha^{n(n-1)} \end{pmatrix}.$$

As α is non-zero and not a root of unity, $\det C(\eta, \tau) \neq 0$. Hence, for $0 \leq d \leq n$, $\sigma(f_d) = 0$, that is, $f_d \in tS$.

By (39) it follows that $y^n x^d f_d \in M$ for $d < n$, whence $0 \equiv y^n b \equiv y^n x^n f_n \bmod M$. By (37),

$$0 \equiv (\alpha^{n-1}\psi + \tau)(\alpha^{n-2}\psi + \tau)\cdots(\psi + \tau) f_n \bmod M. \tag{42}$$

Now, for $0 \leq j \leq n-1$, $\alpha^j \psi + \tau = \alpha^j(\eta - \frac{\alpha\gamma}{1-\alpha}) + \frac{\alpha\gamma}{1-\alpha} = \alpha^j \eta + \gamma[j+1]_\alpha \neq 0$. Hence $f_n \in M \cap tS = (t-\eta)S \cap tS = ((t-\eta)\mathbb{C}[t] \cap t\mathbb{C}[t])S$. As $\eta \neq 0$, $f_n \in t(t-\eta)S$, contradicting the choice of b. This completes the proof. ∎

THEOREM 9.2. *Let $R = A(\alpha, 0, \gamma)$ and suppose that α is non-zero and not a root of unity. Let P be a non-zero prime ideal of R. Then P contains one of the following prime ideals*:

(i) RtR,

(ii) zR (*if* $\gamma \neq 0$),

(iii) $Rt(t-\eta)R$ *where* $\eta \neq 0$, $\eta \neq \frac{\alpha\gamma}{1-\alpha}$, *and* $\alpha^{d-1}\eta + [d]_\alpha \gamma \neq 0$ *for all* $d \geq 1$,

(iv) $Q(M_\eta)$, *where* $\alpha^{d-1}\eta + [d]_\alpha \gamma = 0$ *for some* $d \geq 1$ (*if* $\gamma \neq 0$).

Proof. Suppose that $RtR \not\subseteq P$. By Theorem 7.10 and Proposition 7.13, $Rt(t-\eta)R \subseteq P$ for some non-zero $\eta \in \mathbb{C}$. If $\eta \neq \frac{\alpha\gamma}{1-\alpha}$ and $\alpha^{d-1}\eta + [d]_\alpha \neq 0$ for all $d \geq 1$ then, by Theorem 9.1, $Rt(t-\eta)R$ is the annihilator of the simple module $V(M_\eta)$ and so is prime.

Suppose that $\eta = \frac{\alpha\gamma}{1-\alpha}$. A simple calculation shows that $\alpha^{d-1}\eta + [d]_\alpha\gamma \neq 0$ for all $d \geq 1$. From Remark 7.12, $tz = t^2 - \eta t \in P$ so, as z is normal and $t \notin P$, $zR \subseteq P$. If $\gamma \neq 0$ then zR is prime by Corollary 7.7. Suppose that $\gamma = 0$. As $t = yx - \alpha xy \notin P$, $x \notin P$. By the proof of Corollary 7.7, $xyRx \subseteq P$ so $xy \in P$, whence $t = z - \alpha xy \in P$. This completes the proof in the case $\gamma = 0$.

The remaining possibility is that $\gamma \neq 0$ and $\alpha^{d-1}\eta + [d]_\alpha\gamma = 0$ for some (unique) $d \geq 1$. In this case $V(M_\eta)$ has length two and is annihilated by $Q(M_\eta)t$. Thus $Q(M_\eta)t \subseteq M_\eta R + xR$, and, as $tx = 0$, $tQ(M_\eta)t \subseteq t(t-\eta)R \subseteq P$. As $t \notin P$ it follows that $Q(M_\eta) \subseteq P$. ∎

It is a routine matter to check that the only inclusions between the primes listed in Theorem 9.2 are given by $Rt(t-\eta)R \subset RtR$. Hence we have the following corollary.

COROLLARY 9.3. *Let $R = A(\alpha, 0, \gamma)$ and suppose that α is non-zero and not a root of unity. If $\gamma \neq 0$ then the height one primes of R are as listed in Theorem* 9.2 (ii)–(iv) *and if $\gamma = 0$ they are the ideals $Rt(t-\eta)R$, $\eta \in \mathbb{C}^*$.*

To complete the analysis of the prime spectrum of $A(\alpha, 0, \gamma)$, it now suffices to determine the prime ideals for each of the factors R/P where P is one of the height one primes. For those of the form $Rt(t-\eta)R$, Proposition 7.5 is applicable.

COROLLARY 9.4. *Let $R = A(\alpha, 0, \gamma)$ and suppose that α is non-zero and not a root of unity. Let $\eta \in \mathbb{C}$. If $\eta \neq 0$, $\eta \neq \frac{\alpha\gamma}{1-\alpha}$, and $\alpha^{d-1}\eta + [d]_\alpha\gamma \neq 0$ for all $d \geq 1$ then $R/Rt(t-\eta)R$ has a unique minimal non-zero ideal $RtR/Rt(t-\eta)R$.*

Proof. Let N be the maximal right ideal $(t-\eta)R + xR$. Then $tN \subseteq Rt(t-\eta)R = \operatorname{ann}(R/N)$ by Theorem 9.1. The result follows from Proposition 7.5. ∎

Remark 9.5. Suppose that α is non-zero and not a root of unity and let $R = A(\alpha, 0, \gamma)$. If $\eta = \frac{\alpha\gamma}{1-\alpha}$ then a simple calculation shows that $c_d \notin M_\eta$ for all $d \geq 1$ whence the Verma module $V(M_\eta)$ is simple. It can be checked that the annihilator of $V(M_\eta)$ is zR. By Corollary 7.7, the factor R/zR again has a unique minimal non-zero ideal.

The next result is a consequence of Corollaries 9.3, 9.4, and 7.7, Proposition 7.8, and the descriptions of the prime spectra of A_1^q and $C(q)$ in Section 2.6.

THEOREM 9.6. *Let $R = A(\alpha, 0, \gamma)$ and suppose that α is non-zero and not a root of unity. If $\gamma \neq 0$ then the prime ideals of R are*:

(i) *the height one primes listed in Theorem* 9.2 (ii)–(iv),

(ii) *a unique height two prime RtR*,

(iii) *a unique height three prime $zR + tR$*,

(iv) *the ideals of the form $zR + tR + (y - \tau)R$, $\tau \in \mathbb{C}^*$.*

If $\gamma = 0$ then the prime ideals of R are:

(i) *the height one primes given in Theorem* 9.2(iii),

(ii) *a unique height two prime RtR*,

(iii) *two height three primes $RtR + RyR$ and $RtR + RxR$*,

(iv) *the ideals of the forms $RtR + (x - \tau)R + (y - \psi)R$, where $\tau, \psi \in \mathbb{C}$ and $\tau\psi = 0$.*

9.2. *The case $\alpha = 1$*

Here we assume that $\alpha = 1$ and $\gamma \neq 0$. Without loss of generality, $\gamma = 1$. In this case each $c_d = t + d$ and, for each $d \geq 1$, there is a d-dimensional simple module $L(M_\eta)$, where $\eta = -d$ and $M_\eta = (t - \eta)\mathbb{C}[t]$.

THEOREM 9.7. *Let $R = A(1, 0, 1)$. For $\eta \in \mathbb{C}$, let M_η be the maximal ideal $(t - \eta)\mathbb{C}[t]$. Suppose that $\eta + d \neq 0$ for all integers $d \geq 0$. Then the annihilator of the Verma module $V(M_\eta)$ is $Rt(t - \eta)R$.*

Proof. The structure of the proof and our notation are the same as for the proof of Theorem 9.1 and we indicate how the details differ. The condition $\eta + d \neq 0$ ensures that $V(M_\eta)$ is simple. As before $Rt(t - \eta)R \subseteq \operatorname{ann} V(M_\eta)$ and, if the reverse inclusion is false, we can choose $b = \sum_{i=0}^{n} x^i f_i \in \operatorname{ann} V(M_\eta)$, where each $f_i \in S$ and $f_n \notin t(t - \eta)S$.
The equation (7) gives

$$y^d x^d = (xy + t + d)(xy + t + d - 1) \cdots (xy + t + 1) \tag{43}$$

$$\text{so} \quad y^d x^d f_d \equiv (\eta + d)(\eta + d - 1) \cdots (\eta + 1) f_d \bmod M. \tag{44}$$

Now $y(xy + t + i) = (xy + t + i + 1)y$ and σ is idempotent so, for $j \geq 1$,

$$y^{d+j} x^d f_d = (xy + t + d + j)(xy + t + d + j - 1) \cdots (xy + t + 1 + j)\sigma(f_d) y^j$$

and, modulo M,

$$y^{d+j} x^d f_d \equiv (\eta + d + j)(\eta + d + j - 1) \cdots (\eta + 1 + j)\sigma(f_d) y^j. \tag{45}$$

Let $0 \le i, d \le n$ and let $m = n + i$. It follows from (45) that, modulo M,

$$0 \equiv y^{n+i+1}\left(\sum_{d=0}^{n} x^d f_d\right) y^{n-i-1}$$

$$\equiv \sum_{d=0}^{n} (\eta + m + 1)(\eta + m) \cdots (\eta + m + 2 - d)\sigma(f_d) y^{2n-d}.$$

Thus $C_{n+1}(\eta)V = 0$ where $C_{n+1}(\eta)$ is the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & \eta + n + 1 & (\eta + n + 1)(\eta + n) & \dots & \prod_{s=1}^{n}(\eta + s + 1) \\ 1 & \eta + n + 2 & (\eta + n + 2)(\eta + n + 1) & \dots & \prod_{s=2}^{n+1}(\eta + s + 1) \\ 1 & \eta + n + 3 & (\eta + n + 3)(\eta + n + 2) & \dots & \prod_{s=3}^{n+2}(\eta + s + 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \eta + 2n + 1 & (\eta + 2n + 1)(\eta + 2n) & \dots & \prod_{s=n+1}^{2n}(\eta + s + 1) \end{pmatrix},$$

with entries in $\mathbb{C}$ and V is as before. Successively subtracting row j from row $j + 1$ for $j = n, n - 1, \dots, 1$, we see that

$$\det C_{n+1}(\eta) = n! \det C_n(\eta + 1).$$

As $\det C_1(\eta) = 1$, it follows that

$$\det C_{n+1}(\eta) = n!(n-1)! \cdots 2! \neq 0.$$

Hence each $\sigma(f_d) = 0$ and each $f_d \in tS$.

By (45), $y^n x^d f_d \equiv 0$ for $d < n$ so, by (44),

$$0 \equiv y^n b \equiv y^n x^n f_n \equiv (\eta + n)(\eta + n - 1) \cdots (\eta + 1) f_n. \tag{46}$$

By hypothesis on η, $(\eta + n)(\eta + n - 1) \cdots (\eta + 1) \neq 0$ so $f_n \in t(t - \eta)S$, contradicting the choice of b. ∎

Remark 9.8. Theorem 9.7 is no longer true if $\gamma = 0$, in which case $xy^2 + (t - \eta)y \in \operatorname{ann} V(M_\eta)$.

THEOREM 9.9. *Let $R = A(1, 0, 1)$. Let P be a non-zero prime ideal of R. Then P contains one of the following prime ideals*:

(i) RtR,

(ii) $Rt(t - \eta)R$, *where* $\eta \neq -d$ *for all* $d \ge 0$,

(iii) $Q(M_\eta)$, *where* $\eta = -d$ *for some* $d \ge 1$.

Proof. The proof is similar to that of Theorem 9.2. ∎

COROLLARY 9.10. *Let* $R = A(1, 0, 1)$. *The height one primes of* R *are as listed in Theorem* 9.9 (ii) *and* (iii).

COROLLARY 9.11. *Let* $R = A(1, 0, 1)$. *If* $\eta \neq -d$ *for all* $d \geq 0$, *then* $R/Rt(t-\eta)R$ *has a unique minimal non-zero ideal* $RtR/Rt(t-\eta)R$.

Proof. The proof is similar to that of Corollary 9.4. ∎

THEOREM 9.12. *Let* $R = A(1, 0, 1)$. *The prime ideals of* R *are*:

(i) *the height one primes of* R *listed in Theorem* 9.9 (ii) *and* (iii),

(ii) *a unique height two prime* RtR.

Proof. Given that $R/RtR \simeq A_1(\mathbb{C})$ is simple, this follows easily from Corollaries 9.10 and 9.11. ∎

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